Astrophysics III **Astrophysical Fluid Dynamics**

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Contents

References

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1 Basic Equations of Fluid Mechanics

1.1 What is fluid mechanics?

- Fluid dynamics is the study of finding the velocity field $\boldsymbol{v}(\boldsymbol{x}, t)$, the density field $\rho(\boldsymbol{x}, t)$, and the pressure field $p(\boldsymbol{x}, t)$.

- Conditions for the continuum approximation
  The size of the system (i.e., characteristic length over which the velocity field varies) should be much longer than the **mean free path** l or the molecular diameter d ($\sim 10^{-10}$m).
  $$l \simeq \frac{1}{n\pi d^2} \sim 10^{13} \left(\frac{n}{10^6 \mathrm{m}^{-3}}\right)^{-1} \mathrm{m}.$$
  For example, in the interstellar medium, the number density of gas molecules n is about $10^6 \mathrm{m}^{-3}$ and the mean free path l is 10^{-3}pc.

1.2 Equation of continuity and mass conservation

Consider the mass of fluid entering and leaving from a given volume V due to a flow.

- The mass of fluid leaving through a small surface, dS, which is part of the surface S of the volume, during a time inteval Δt can be written as (see Figure 1)
  $$\text{Outgoing mass} \quad = \quad \text{density} \times \text{outgoing volume} \quad = \quad \rho\,(\Delta t\, \boldsymbol{v} \cdot \boldsymbol{n}\, dS), \qquad (1.1)$$
  where $\boldsymbol{n}$ is the outward normal vector of the surface dS. Therefore, the total mass of fluid leaving the volume V in time Δt is
  $$\Delta t \oint_S \rho \boldsymbol{v} \cdot \boldsymbol{n}\, dS = \Delta t \int_V \nabla \cdot (\rho \boldsymbol{v})\, dV \qquad \text{(Gauss' theorem)}. \qquad (1.2)$$

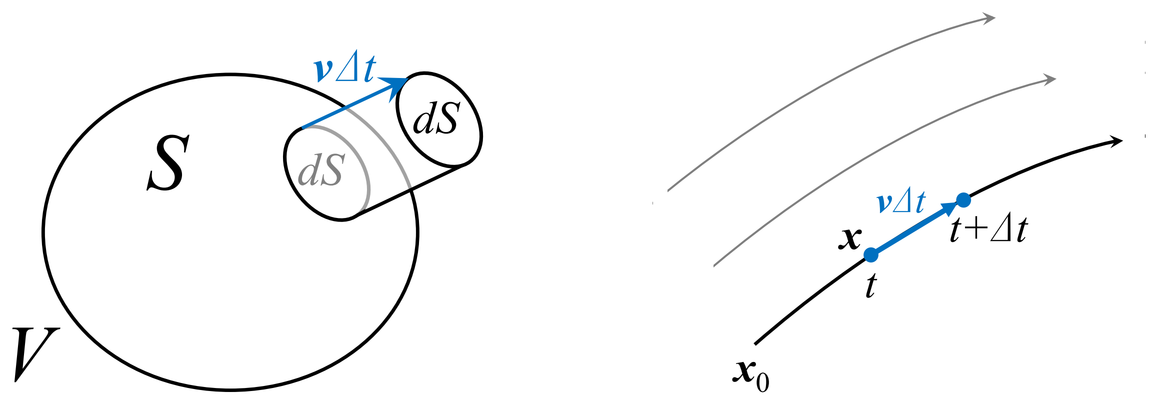


Figure 1: Left: Flow leaving from small part dS of the surface of the volume V. Right: Streamlines and motion of a fluid particle.

- Using Eq. (1.2), the **mass conservation equation** can be written as

$$\frac{d}{dt}\int_V \rho\, dV = -\int_V \nabla\cdot(\rho\boldsymbol{v})\, dV. \tag{1.3}$$

This means that the decrease in mass within V equals the mass of the outgoing fluid. Combining both sides into one yields

$$\int_V \left[\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\boldsymbol{v})\right] dV = 0. \tag{1.4}$$

Since this equation holds for any V, the integrand must vanishes anywhere. Therefore

$$\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\boldsymbol{v}) = 0. \qquad \left(\text{or}\quad \frac{\partial\rho}{\partial t} + \text{ div}(\rho\boldsymbol{v}) = 0.\right) \tag{1.5}$$

This equation is called the **equation of continuity** and describes the time evolution of the density of the fluid.

- The vector $\boldsymbol{j} = \rho\boldsymbol{v}$ in the divergence in the above equation is called the **mass flux density (vector)**. The direction of the vector indicates the direction of the flow, and the length represents the mass of fluid passing through unit area perpendicular to the flow per unit time. The **mass flux**[1] is the surface integral of the mass flux density as shown on the left side of Eq. (1.2). Since Eq. (1.4) indicates mass conservation, the equation of continuity (1.5) is also called the equation of mass conservation.

1.3 Euler's equation and momentum conservation

(a) Acceleration of fluid

- Particle acceleration

$$\frac{d\boldsymbol{v}}{dt} \equiv \lim_{\Delta t\to 0}\frac{\boldsymbol{v}(t+\Delta t) - \boldsymbol{v}(t)}{\Delta t}. \tag{1.6}$$

It is defined by the difference in velocity of the same particle at different times.

- The time partial derivative of velocity at position $\boldsymbol{x}$, which is used in fluid mechanics, is

$$\frac{\partial\boldsymbol{v}}{\partial t} \equiv \lim_{\Delta t\to 0}\frac{\boldsymbol{v}(\boldsymbol{x}, t+\Delta t) - \boldsymbol{v}(\boldsymbol{x}, t)}{\Delta t}. \tag{1.7}$$

This is not an "acceleration" because it is not a difference in velocity with respect to the same fluid particle.

[1] In general, fluxes indicate flow rates across a surface. There are various types of fluxes depending on what is flowing, such as the energy flux, momentum flux, heat flux, and charge current, in addition to the mass flux. In electromagnetism, electric and magnetic fluxes are another type of fluxes.

- Expression for the acceleration of a fluid particle

  To obtain the acceleration, we take the difference of the velocities along its trajectory $\boldsymbol{x}(t, \boldsymbol{x}_0)$ of the fluid particle (see the right panel of Fig. 1, $\boldsymbol{x}_0$ is the initial position of the fluid particle).

$$\frac{D\boldsymbol{v}}{Dt} \equiv \lim_{\Delta t \to 0} \frac{\boldsymbol{v}(\boldsymbol{x} + \boldsymbol{v}\Delta t,\ t + \Delta t) - \boldsymbol{v}(\boldsymbol{x}, t)}{\Delta t}. \tag{1.8}$$

  The time derivative defined along the flow is called the **Lagrangian derivative**. Furthermore, the velocity $\boldsymbol{v}(\boldsymbol{x} + \boldsymbol{v}\Delta t,\ t + \Delta t)$ is rewritten in the Taylor expansion. In the one-dimensional case,

$$v(x + v\Delta t,\ t + \Delta t) = v(x, t) + \frac{\partial v}{\partial t}\Delta t + v\frac{\partial v}{\partial x}\Delta t. \tag{1.9}$$

  and in the three-dimensional case

$$\boldsymbol{v}(\boldsymbol{x} + \boldsymbol{v}\Delta t,\ t + \Delta t) = \boldsymbol{v}(\boldsymbol{x}, t) + \frac{\partial \boldsymbol{v}}{\partial t}\Delta t + (\boldsymbol{v} \cdot \nabla)\boldsymbol{v}\Delta t, \tag{1.10}$$

  or

$$\boldsymbol{v}(\boldsymbol{x} + \boldsymbol{v}\Delta t,\ t + \Delta t) = \boldsymbol{v}(\boldsymbol{x}, t) + \frac{\partial \boldsymbol{v}}{\partial t}\Delta t + (\boldsymbol{v} \cdot \mathbf{grad})\boldsymbol{v}\Delta t.$$

  Therefore, substituting Eq. (1.10) into (1.8), we obtain the expression for the acceleration of fluid as

$$\frac{D\boldsymbol{v}}{Dt} = \frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v} \cdot \nabla)\boldsymbol{v} = \frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v} \cdot \mathbf{grad})\boldsymbol{v}. \tag{1.11}$$

  The expression for the acceleration on the right side is called the **Eulerian form**.

(b) Euler's equation

The force exerted by the pressure on a fluid particle is considered next. The force acting on the volume V is derived in the same way as in the equation of continuity. The force exerted on a small surface dS by the external pressure p is $-p\boldsymbol{n}\, dS$. Thus, the total force due to the external pressure acting on the whole surface of the volume V is given by

$$-\oint_S p\boldsymbol{n}\, dS = -\int_V \nabla p\, dV. \tag{1.12}$$

Problem 1. Derive Eq. (1.12) using Gauss's divergence theorem.

We can now write down the equation of motion. Let us consider the conservation of the momentum for a volume V. Assume that no force other than pressure acts on the volume. Let us also assume that the surface S of V moves with the fluid and that no fluid flows into or out of the volume. In this case, the conservation of the momentum for the fluid in the volume V is expressed as

$$\int_V \rho \frac{D\boldsymbol{v}}{Dt}\, dV = -\int_V \nabla p\, dV. \tag{1.13}$$

Thus, we finally obtain the equations of motion in fluid mechanics, the so-called **Euler's equation**, as

$$\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v} \cdot \nabla)\boldsymbol{v} = -\frac{1}{\rho}\nabla p. \qquad \text{or} \qquad \frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v} \cdot \mathbf{grad})\boldsymbol{v} = -\frac{1}{\rho}\nabla p. \tag{1.14}$$

Euler's equation describes the time evolution of the velocity of fluid. If there is an external force $\boldsymbol{f}$ acting on the unit mass of the fluid other than pressure (such as gravity), we can add it to the right-hand side

$$\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v} \cdot \nabla)\boldsymbol{v} = -\frac{1}{\rho}\nabla p + \boldsymbol{f}. \tag{1.15}$$

(c) Conservation form of Euler's equation

- The time variation of the momentum per unit (fixed) volume is

$$\frac{\partial}{\partial t}(\rho \boldsymbol{v}) = \boldsymbol{v}\frac{\partial \rho}{\partial t} + \rho\frac{\partial \boldsymbol{v}}{\partial t} = -\boldsymbol{v}\nabla \cdot (\rho \boldsymbol{v}) - \rho(\boldsymbol{v} \cdot \nabla)\boldsymbol{v} - \nabla p. \tag{1.16}$$

  In the second equality, we used the equation of continuity (1.5) and Euler's equation (1.14). Let us rewrite this equation using the components v_i of the vector $\boldsymbol{v}$. The subscript i is 1, 2, or 3, indicating the x, y, and z components, respectively. Then, Eq. (1.16) is rewritten as

$$\frac{\partial}{\partial t}(\rho v_i) = -v_i\frac{\partial}{\partial x_j}(\rho v_j) - \rho v_j\frac{\partial v_i}{\partial x_j} - \frac{\partial p}{\partial x_i}. \tag{1.17}$$

  where the subscript j also takes 1, 2, and 3. The product of components with the same subscript $A_j B_j$ is assumed to take a sum from 1 to 3 (Einstein notation) and thus equal to the scalar product $\boldsymbol{A} \cdot \boldsymbol{B}$. Such a sum is also taken in Eq. (1.17).

- Equation (1.17) is eventually written in the following form (**conservation form of Euler's equation**)

$$\frac{\partial}{\partial t}(\rho v_i) + \frac{\partial}{\partial x_j}\Pi_{ij} = 0, \tag{1.18}$$

  where the tensor Π_{ij} is given by

$$\Pi_{ij} = \delta_{ij} p + \rho v_i v_j, \tag{1.19}$$

  using Kronecker's delta δ_{ij}, and called the **momentum flux density tensor**. The first term of this tensor represents the momentum flow across the surface caused by the pressure acting on the surface, and the second term represents the momentum flow due to the fluid directly passing through the surface.

- Integrating Eq. (1.18) over a volume V with the fixed surface S, we obtain the integral form of the **momentum conservation equation** as

$$\frac{d}{dt}\int_V \rho v_i \, dV = \int_V \frac{\partial}{\partial t}(\rho v_i)\, dV = -\oint_S \Pi_{ij} n_j \, dS. \tag{1.20}$$

  The right-hand side represents the total momentum flowing out of the surface S. Thus, the tensor Π_{ij} represents the flow of the i-component of the momentum across the surface perpendicular to the basis vector $\boldsymbol{e}_j$.

Problem 2. For the volume V that deforms together with the flow used in Eq. (1.13), clarify whether or not the following equation holds.

$$\frac{d}{dt}\int_V \rho \boldsymbol{v}\, dV = \int_V \rho \frac{D\boldsymbol{v}}{Dt}\, dV \tag{1.21}$$

(Hint: Instead of integrating by volume, consider summing up the infinitesimal mass elements $dM = \rho dV$.)

(d) Additional remarks on Euler's equation

- Viscous effects may be important in real fluids, but they are not considered in Euler's equation. A fluid whose viscosity is not important is called an **ideal fluid**. For many flows in the universe, the ideal fluid approximation is valid. The equation of motion that takes viscous effects into account is called the **Navier-Stokes equation**.

- **Boundary conditions:** If there is an object in the fluid, conditions are required at its surface (boundary). In an ideal fluid, it is required that the fluid does not penetrate the inside of the object. Therefore, the boundary condition at the surface of the object is given by

$$v_n = \boldsymbol{v}\cdot\boldsymbol{n} = \boldsymbol{u}\cdot\boldsymbol{n} \tag{1.22}$$

  where $\boldsymbol{n}$ is the normal vector of the surface and $\boldsymbol{u}$ is the velocity of the object (surface). In an ideal fluid, the tangential velocity along the surface may be different between the fluid and the object.

1.4 Condition for an adiabatic flow

- There are two independent thermodynamic quantities (e.g., pressure and density). Therefore, in addition to the equation of continuity, we need another equation to determine the time evolution of the thermodynamic quantities. In an ideal fluid, the invariance of the entropy (or an **adiabatic flow**) is often assumed. In other words, the entropy s per unit mass is assumed to be invariant for "each fluid particle". This is expressed using the Lagrangian derivative as

$$\frac{Ds}{Dt} = \frac{\partial s}{\partial t} + \boldsymbol{v}\cdot\nabla s = 0 \tag{1.23}$$

This is the condition for an adiabatic flow[2].

- A simpler case is often set that the entropy is constant over the entire fluid region at a certain time. In this case, the entropy remains constant thereafter due to the adiabatic condition.

$$s = \text{const.} \tag{1.24}$$

Such a flow is called an **isentropic flow**.

- In the adiabatic process of an ideal gas, each fluid particle satisfies the relation

$$p = K\rho^{\gamma}, \tag{1.25}$$

where $\gamma = c_p/c_V$ is the specific heat ratio, and the coefficient K is a function of the entropy s of each fluid particle (see Eq. [1.65]). Since K is constant throughout the fluid for an isentropic flow, this equation is often used for an isentropic flow. If there is heating or cooling due to radiation, the adiabatic condition or Eq. (1.25) does not hold. Even in such cases, however, Eq. (1.25) may be used approximately using another constant Γ instead of the specific heat ratio γ. In those cases, Eq. (1.25) is called the **polytropic relation** and Γ is the **polytropic index**.

- In an isentropic flow, the differential of the enthalpy per unit mass, h, is given by

$$dh = Tds + Vdp = \frac{1}{\rho}dp, \tag{1.26}$$

where $V = 1/\rho$ is the specific volume. From this, in an isentropic flow, Euler's equation (1.14) can be written as

$$\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{v} = -\nabla h. \tag{1.27}$$

- Even in the case of a non-isentropic flow, if the pressure p is a function of only the density ρ, as in the polytropic relation (1.25), Eq. (1.27) can be satisfied, by defining $h \equiv \int(1/\rho)dp$. In general, a fluid in which p is a function of only ρ is called a **barotropic fluid** (otherwise it is called a **baroclinic fluid**).

1.5 Bernoulli's equation

- Consider a time-independent flow (steady flow) in which the adiabatic condition holds[3]. In steady flow, $\partial\boldsymbol{v}/\partial t = 0$. In addition to this, using the vector formula

$$(\boldsymbol{v}\cdot\nabla)\boldsymbol{v} = \frac{1}{2}\nabla v^2 - \boldsymbol{v}\times(\nabla\times\boldsymbol{v}) \qquad \text{or} \qquad v_j\frac{\partial v_i}{\partial x_j} = \frac{1}{2}\frac{\partial v^2}{\partial x_i} - \epsilon_{ijk}v_j\epsilon_{klm}\frac{\partial v_m}{\partial x_l} \tag{1.28}$$

[2] From this adiabatic condition and the equation of continuity (1.5), we also obtain **entropy conservation equation** as $\partial(s\rho)/\partial t + \nabla\cdot(s\rho\boldsymbol{v}) = 0$.

[3] Even in a barotropic fluid, where the pressure p is a function of only the density, by using $h = \int(1/\rho)dp$, Eq. (1.27) is satisfied, leading to Eq. (1.30) and and Bernoulli's equation (1.31).

(where ϵ_{ijk} is the Levi-Civita symbol), Euler's equation (1.14) becomes

$$\frac{1}{2}\nabla v^2 + \frac{1}{\rho}\nabla p - \boldsymbol{v} \times (\nabla \times \boldsymbol{v}) = 0. \tag{1.29}$$

Furthermore, taking the scalar product of this equation and $\boldsymbol{v}$, and noting that the third term on the left side is perpendicular to $\boldsymbol{v}$, we obtain

$$\frac{1}{2}\boldsymbol{v} \cdot \nabla v^2 + \frac{1}{\rho}\boldsymbol{v} \cdot \nabla p = \boldsymbol{v} \cdot \nabla \left(\frac{1}{2}v^2 + h\right) = 0. \tag{1.30}$$

In the first equality of the above equation, we used the relation $\nabla h = T\nabla s + \frac{1}{\rho}\nabla p$, which is derived from $dh = Tds + \frac{1}{\rho}dp$, and the adiabatic condition for steady flow $\boldsymbol{v} \cdot \nabla s = 0$.

- Consider a streamline. A streamline is a line tangent to a velocity vector at each point on the line. In a steady flow, the streamlines represent the trajectories of fluid particles. Thus, the left and middle sides of Eq. (1.30) represent the gradients along the streamlines. Therefore, the following equation is satisfied along each streamline.

$$\frac{1}{2}v^2 + h = \text{const.} \tag{1.31}$$

In general, this constant is different for each streamline. Equation (1.31) is called **Bernoulli's equation**. Bernoulli's equation shows that the velocity increases as the pressure decreases and that the pressure has its maximum at the point where the velocity vanishes (i.e., at the **stagnation point**).

- A similar equation is derived for a fluid in a gravitational field. In this case we use Euler's equation (1.15) including the external force. Using the gravitational potential ϕ_g, the external force per unit mass is given by $\boldsymbol{f} = -\nabla\phi_g$, and thus Bernoulli's equation in the gravitational field becomes

$$\frac{1}{2}v^2 + h + \phi_g = \text{const.} \tag{1.32}$$

Problem 3. Derive the vector formula (1.28).

1.6 Energy conservation equation

- The energy per unit volume of a fluid is given by the sum of its kinetic energy and internal energy, $\frac{1}{2}\rho v^2 + \rho e$, where e is the internal energy per unit mass. Consider the time variation of the energy of the fluid. The adiabatic condition (1.23) is assumed to hold.

- The time derivative of the kinetic energy becomes

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho v^2\right) = \frac{1}{2}v^2\frac{\partial \rho}{\partial t} + \rho v_i \frac{\partial v_i}{\partial t}. \tag{1.33}$$

Furthermore, using the equation of continuity and Euler's equation, we obtain

$$\begin{aligned}\frac{\partial}{\partial t}\left(\frac{1}{2}\rho v^2\right) &= -\frac{1}{2}v^2\frac{\partial}{\partial x_i}(\rho v_i) - \rho v_i v_j \frac{\partial v_i}{\partial x_j} - v_i \frac{\partial p}{\partial x_i} \\ &= -\frac{1}{2}v^2\frac{\partial}{\partial x_i}(\rho v_i) - \rho v_j \frac{\partial}{\partial x_j}\left(\frac{1}{2}v^2 + h\right) + \rho v_i T \frac{\partial s}{\partial x_i}. \quad (1.34)\end{aligned}$$

In the above, we used $\frac{1}{\rho}\nabla p = -T\nabla s + \nabla h$.

- Next, the time derivative of the internal energy is

$$\begin{aligned}\frac{\partial}{\partial t}(\rho e) &= \rho\frac{\partial e}{\partial t} + e\frac{\partial \rho}{\partial t} = \rho T\frac{\partial s}{\partial t} + h\frac{\partial \rho}{\partial t} \\ &= -\rho v_i T\frac{\partial s}{\partial x_i} - h\frac{\partial}{\partial x_i}(\rho v_i). \quad (1.35)\end{aligned}$$

  In the above, we used $de = Tds + (p/\rho^2)d\rho$ and $h = e + p/\rho$ for the second equality. The adiabatic condition and the equation of continuity are used for the third equality.

- Adding Eq. (1.34) and (1.35), we obtain

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho v^2 + \rho e\right) = -\left(\frac{1}{2}v^2 + h\right)\frac{\partial}{\partial x_i}(\rho v_i) - \rho v_i \frac{\partial}{\partial x_i}\left(\frac{1}{2}v^2 + h\right). \quad (1.36)$$

  Putting the terms on the right-hand side together, we finally obtain the **energy conservation equation**.

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho v^2 + \rho e\right) + \mathrm{div}\left[\rho \boldsymbol{v}\left(\frac{1}{2}v^2 + h\right)\right] = 0. \quad (1.37)$$

  The vector in the divergence is called the **energy flux density vector**.

- In steady flow, both the mass flux ($\rho v_i dS_i$) and the energy flux through the flow tube, which is surrounded by streamlines, are constant from the energy conservation equation (1.37) and the equation of continuity (1.5), respectively, where dS_i is the cross section of the tube. The ratio of these fluxes is also constant, yielding Bernoulli's equation (1.31).

- Additional remarks
  - When an external force $\boldsymbol{f}$ acts, the power exerted by the external force, $\rho\boldsymbol{v}\cdot\boldsymbol{f}$, is added to the right-hand side of Eq. (1.37).
  - If there is heating (positive or negative) due to radiation absorption/emission or chemical reactions, a source term for the heating rate per unit time and unit volume must be added to the right-hand side of Eq. (1.37). In addition to Eq. (1.37), the equation for energy transport by radiation etc. is also needed.

- In general, the conservation form of the hydrodynamic equations (and other physical equations) is expressed in the form

$$\frac{\partial}{\partial t}(\text{... density}) + \operatorname{div}(\text{... flux density}) = 0. \tag{1.38}$$

Problem 4. Derive the extended expression of the energy conservation equation (1.37) for the case where there is a static external gravitational field $\phi_g(\boldsymbol{r})$, and also find the expression for the energy flux density vector. Note that the energy per unit volume of fluid in an external gravitational field is $\frac{1}{2}\rho v^2 + \rho e + \rho\phi_g$.

1.7 Incompressible fluid and its potential flow

- In an incompressible fluid, the density ρ is approximately constant. Liquids can be considered to be incompressible fluids. More specifically, the condition for the incompressible fluid approximation to hold is $v \ll c_s$, where c_s is the velocity of sound. Furthermore, for the incompressible fluid approximation to hold, the variation time T of the flow must also be long enough as $T \gg L/c_s$, where L is the characteristic length of the flow. Therefore, the incompressible fluid approximation is also valid for subsonic gaseous flows.

- Basic equations for incompressible fluids are

$$\text{Eq. of continuity:} \qquad \nabla \cdot \boldsymbol{v} = 0, \tag{1.39}$$

$$\text{Euler's equation:} \qquad \frac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{v} \cdot \nabla \boldsymbol{v} = -\nabla\left(\frac{p}{\rho}\right). \tag{1.40}$$

- **Potential flow of an incompressible fluid**: When the velocity is expressed with a potential $\phi(\boldsymbol{r}, t)$ as

$$\boldsymbol{v} = \mathbf{grad}\,\phi, \tag{1.41}$$

the flow is called a **potential flow**. Potential flow is "irrotational" because the vortex is written as

$$\operatorname{rot}\boldsymbol{v} = \nabla \times (\nabla\phi) = 0. \tag{1.42}$$

The equation governing a potential flow is obtained from Eq. (1.39) as

$$\triangle\phi = 0 \qquad \text{(Laplace's equation)}. \tag{1.43}$$

Furthermore, using Eqs. (1.28) and (1.42), Euler's equation (1.40) is rewritten as

$$\nabla\left(\frac{\partial\phi}{\partial t} + \frac{1}{2}v^2 + \frac{p}{\rho}\right) = 0, \tag{1.44}$$

and we obtain

$$\frac{\partial\phi}{\partial t} + \frac{1}{2}v^2 + \frac{p}{\rho} = \text{constant}. \tag{1.45}$$

An example of incompressible potential flow: "Flow around a sphere"

Consider a potential flow of incompressible ideal fluid around a rigid sphere of radius R at rest. Away from the sphere, the flow is uniform, $\boldsymbol{v} = \boldsymbol{u}$ (constant).

- We use a coordinate system in which the origin is at the center of the sphere and the z-axis is in the direction of the flow. By symmetry, the flow (and the potential ϕ) is axisymmetric about the z-axis.
- Boundary conditions. At the surface of the sphere ($r = R$), $v_r = 0$, or

$$\frac{\partial \phi}{\partial r} = 0 \qquad \text{at } r = R. \tag{1.46}$$

Far away from the sphere, $\boldsymbol{v} = \boldsymbol{u}$. Then, using $u = |\boldsymbol{u}|$, we obtain the potential as

$$\phi = \boldsymbol{u} \cdot \boldsymbol{r} = u\, r \cos\theta \qquad \text{at } r \to \infty. \tag{1.47}$$

- In general, axisymmetric solutions of Laplace's equation are expressed with the Legendre polynomials P_l as

$$\phi = \sum_{l=0}^{\infty} \left(A_l\, r^l + \frac{B_l}{r^{l+1}} \right) P_l(\cos\theta). \tag{1.48}$$

From the shape of the system and the boundary conditions, it is expected that only the first two terms in the summation need to be used.

$$\phi = \left(A_0 + \frac{B_0}{r} \right) + \left(A_1\, r + \frac{B_1}{r^2} \right) \cos\theta. \tag{1.49}$$

(The Legendre polynomials are $P_0 = 1, \quad P_1 = x, \quad P_2 = (3x^2 - 1)/2, \quad \ldots$)

- The coefficients, A_i, B_i are determined by the boundary conditions.

  - The condition (1.47) at $r \to \infty$ gives $\quad A_0 = 0, \quad A_1 = u.$

  - The condition (1.46) at $r = R$ is $\quad \dfrac{\partial \phi}{\partial r} = -\dfrac{B_0}{R^2} + \left(u - \dfrac{2B_1}{R^3} \right) \cos\theta = 0.$

  Since this holds for all θ, we obtain $\quad B_0 = 0, \quad B_1 = \frac{1}{2} R^3 u, \quad$ and

$$\phi = \boldsymbol{u} \cdot \boldsymbol{r} \left(1 + \frac{R^3}{2r^3} \right). \tag{1.50}$$

  - The uniqueness of the solution to Laplace's equation assures $A_l = B_l = 0$ for $l \geq 2$.

- The velocity field is given by

$$\boldsymbol{v} \;=\; \boldsymbol{u} + \frac{R^3}{2} \frac{\boldsymbol{u} r^2 - 3(\boldsymbol{u} \cdot \boldsymbol{r})\boldsymbol{r}}{r^5} = \boldsymbol{u} + \frac{R^3}{2r^3}(\boldsymbol{u} - 3u\cos\theta \boldsymbol{e}_r). \tag{1.51}$$

At the surface of the sphere ($r = R$),

$$\boldsymbol{v}(r = R, \theta) = -\frac{3}{2} u \sin\theta \boldsymbol{e}_\theta \tag{1.52}$$

Both poles of the sphere ($\theta = 0$ and π) are stagnation points. Figure 2 shows the stream lines of this flow.

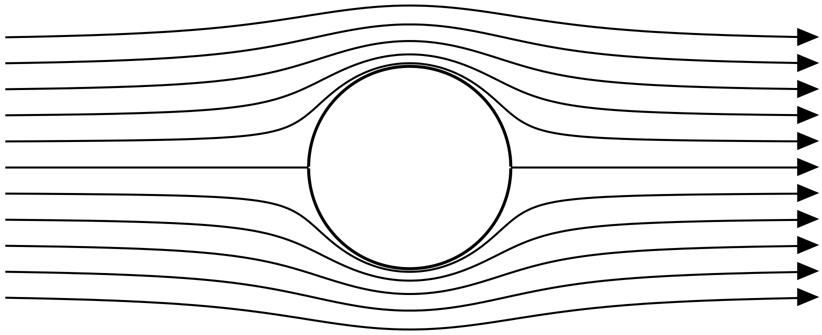


Figure 2: The flow around a sphere. The stream lines obtained from the velocity field are drawn.

- The pressure is obtained from Eq. (1.45). Since this flow is steady, we have

$$\frac{1}{2}v^2 + \frac{p}{\rho} = C. \tag{1.53}$$

  Letting p_0 be the pressure at infinity, we obtain $C = \frac{1}{2}u^2 + p_0/\rho$. Therefore,

$$p(r,\theta) = p_0 + \frac{1}{2}\rho\left[u^2 - v(r,\theta)^2\right]. \tag{1.54}$$

  At the surface of the sphere,

$$p = p_0 + \frac{1}{2}\rho u^2\left(1 - \frac{9}{4}\sin^2\theta\right) \tag{1.55}$$

  and thus the pressure attains its maximum at the stagnation points. Furthermore, since the pressure is symmetrical between the front and back sides ($\theta \gtrless \pi/2$) of the surface (see eq.[1.55]), we find that the drag force on the sphere vanishes (**d'Alembert's paradox**).

Problem 5. Derive Eqs. (1.51), (1.52), and (1.55).

(ASIDE) In order to obtain the realistic drag force on the sphere, we need to include the viscous effect and solve the **Navier-Stoke equation**:

$$\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{v} = -\frac{1}{\rho}\nabla p + \nu \triangle \boldsymbol{v}, \tag{1.56}$$

where ν is the kinematic viscosity. The obtained flow have non-zero vortex (i.e., rot $\boldsymbol{v} \neq 0$). The drag force acting on the sphere is expressed as $F = \frac{1}{2}C_D \pi R^2 \rho u^2$, where the dimensionless coefficient C_D is given by

$$C_D = \begin{cases} 12\nu/(Ru) & \text{for large } \nu \text{ (Stokes' law)}, \\ \sim 1 & \text{for small } \nu. \end{cases} \tag{1.57}$$

1.8 Appendix: Thermodynamics of ideal gas

- **Equation of state**

$$p = \frac{n_{mol}}{V} RT = nk_BT = \frac{\rho}{m} k_BT, \tag{1.58}$$

where n_{mol} is the mole number, $R(= k_BN_A)$ is the molar gas constant, $k_B(= 1.38 \times 10^{-23}[\mathrm{J/K}])$ is the Boltzmann constant, $N_A(= 6.02 \times 10^{23})$ is the Avogadro constant, n is the number density of gas molecules, and m is the mass of a molecule.

- **Specific heat (heat capacity per unit mass)**

- isochoric specific heat (constant V) $\quad c_V = T\left(\frac{ds}{dT}\right)_V = \left(\frac{de}{dT}\right)_V = \frac{1}{\gamma-1}\frac{k_B}{m}. \quad (1.59)$

- isobaric specific heat (constant p) $\quad c_p = T\left(\frac{ds}{dT}\right)_p = \left(\frac{dh}{dT}\right)_p = \frac{\gamma}{\gamma-1}\frac{k_B}{m}. \quad (1.60)$

In the above, the heat capacity ratio γ is given by c_p/c_V. We also used $c_p - c_V = k_B/m$ (Mayer's relation). The specific heat capacities and their ratio γ are assumed to be constants below.

- **Sound velocity**, c_s

$$c_s^2 \equiv \left(\frac{\partial p}{\partial \rho}\right)_s = \gamma \frac{p}{\rho} \qquad (p \propto \rho^\gamma \text{ in an adiabatic process}). \tag{1.61}$$

- **Internal energy** per unit mass, e

$$e = c_VT = \frac{1}{\gamma-1}\frac{k_B}{m}T = \frac{1}{\gamma-1}\frac{p}{\rho} = \frac{c_s^2}{\gamma(\gamma-1)}. \tag{1.62}$$

- **Enthalpy** per unit mass, h $(= e + p/\rho)$

$$h = c_pT = \frac{\gamma}{\gamma-1}\frac{p}{\rho} = \frac{c_s^2}{\gamma-1}. \tag{1.63}$$

- **Entropy** per unit mass, s

$$ds = \frac{1}{T}\left[de + pd\left(\frac{1}{\rho}\right)\right] = c_Vd\ln\left(\frac{p}{\rho^\gamma}\right). \tag{1.64}$$

Therefore[4]

$$s = c_V \ln\left(\frac{p}{\rho^\gamma}\right) + A, \qquad \text{or} \qquad p = A'e^{s/c_V}\rho^\gamma \qquad (A, A' \text{ are constants}). \tag{1.65}$$

[4]Some readers may feel it uneasy that the dimensional quantity, p/ρ^γ, is the argument of the logarithmic function in Eq. (1.65). But, it should be noticed that Eq. (1.65) can be rewritten as the difference from the standard state (p_0, ρ_0, s_0), $s - s_0 = c_V \ln[(p/p_0)/(\rho/\rho_0)^\gamma]$, by setting $A = s_0 - c_V \ln(p_0/\rho_0^\gamma)$.

2 Compressible Fluids

2.1 Sound waves

- **Assumption**
  - Wave amplitude is small enough.
  - Sound waves in a stationary gas with uniform density and temperature.

- **Perturbations**
  Write the pressure, density, and velocity as follows.

$$p = p_0 + p_1, \qquad \rho = \rho_0 + \rho_1, \qquad \boldsymbol{v} = \boldsymbol{v}_1. \tag{2.1}$$

The quantities with subscript 1 are small perturbations and represent sound waves. The quantities with subscript 0 represent the unperturbed state of the background and are constant in this case. For sound waves, the fluctuation time (or the oscillation period) is often so short compared to the heat transfer time that the fluctuations are adiabatic. Therefore, $s_1 = 0$. In such a case, the following equation holds between p_1 and ρ_1.

$$p_1 = \left(\frac{\partial p}{\partial \rho}\right)_s \rho_1. \tag{2.2}$$

The coefficient $(\partial p/\partial \rho)_s$ is given by

$$\left(\frac{\partial p}{\partial \rho}\right)_s = c_s^2 = \gamma \frac{p_0}{\rho_0} = \gamma \frac{k_B T_0}{m}. \tag{2.3}$$

where c_s is the (adiabatic) **sound velocity** and the second and third equalities hold for ideal gases.

- **Wave equation for sound**
  Find the equations for the perturbations. Substituting Eq. (2.1) into the equation of continuity and Euler's equation and neglecting the second- or higher-order terms for small perturbations, we obtain the following two equations.

$$\frac{\partial \rho_1}{\partial t} + \rho_0 \operatorname{div} \boldsymbol{v}_1 = 0, \tag{2.4}$$

$$\frac{\partial \boldsymbol{v}_1}{\partial t} + \frac{1}{\rho_0} \mathbf{grad}\, p_1 = 0. \tag{2.5}$$

Substituting Eq. (2.5) into the time derivative of Eq. (2.4) to eliminate $\boldsymbol{v}_1$, and using Eqs. (2.2) and (2.3) to eliminate p_1, we obtain

$$\frac{\partial^2 \rho_1}{\partial t^2} - c_s^2 \triangle \rho_1 = 0 \tag{2.6}$$

This is the wave equation with the propagation velocity c_s and describes sound waves.

- **The solution for a traveling plane wave** can generally be written as

$$\rho_1 = f(\boldsymbol{k} \cdot \boldsymbol{x} - \omega t), \tag{2.7}$$

where $\omega = c_s|\boldsymbol{k}|$. The plane wave propagates in the direction $\boldsymbol{k}$. From Eqs. (2.5) and (2.2), The velocity perturbation is obtained as

$$\boldsymbol{v}_1 = \frac{\rho_1}{\rho_0} c_s \frac{\boldsymbol{k}}{|\boldsymbol{k}|} \tag{2.8}$$

The sound wave is a **longitudinal wave** because its velocity is in the direction of propagation. In particular, for sine waves given by

$$\rho_1 = A \exp[i(\boldsymbol{k} \cdot \boldsymbol{x} - \omega t)], \tag{2.9}$$

where $\boldsymbol{k}$ is **the wave number vector** and ω is **the angular frequency**. An arbitrary wave is represented by a superposition of sine waves with various $\boldsymbol{k}$.

Problem 6. Derive Eq. (1.64). Also calculate the sound velocity in HI interstellar gas with temperature of 100K.

2.2 Waves with finite amplitudes: simple waves, rarefaction waves, Riemann invariants

Consider **plane waves with finite amplitudes**. We assume isentropic motion ($dp = c_s^2\, d\rho$). For the one-dimensional wave varying in x-direction, the equation of continuity and Euler's equation can be written as.

$$\frac{\partial \rho}{\partial t} + v\frac{\partial \rho}{\partial x} + \rho\frac{\partial v}{\partial x} = 0, \tag{2.10}$$

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} + \frac{c_s^2}{\rho}\frac{\partial \rho}{\partial x} = 0. \tag{2.11}$$

(a) Simple waves

- Moreover, we assume that the density is written as $\rho = \rho(v)$. The waves that satisfy this assumption are called **simple waves**. Then, the equation of continuity (2.10) can be rewritten as

$$\frac{d\rho}{dv}\frac{\partial v}{\partial t} + \left(v\frac{d\rho}{dv} + \rho\right)\frac{\partial v}{\partial x} = 0. \tag{2.12}$$

Dividing this equation by $(d\rho/dv)$ yields the following equation describing the time evolution of v.

$$\frac{\partial v}{\partial t} + \left(v + \rho/\left(\frac{d\rho}{dv}\right)\right)\frac{\partial v}{\partial x} = 0. \tag{2.13}$$

Similarly, Euler's equation (2.11) is transformed as

$$\frac{\partial v}{\partial t} + \left(v + \frac{c_s^2}{\rho}\frac{d\rho}{dv}\right)\frac{\partial v}{\partial x} = 0. \tag{2.14}$$

Since the two differential equations (2.13) and (2.14) should be equal, so

$$\frac{d\rho}{dv} = \frac{\rho}{c_s} \qquad \text{or} \qquad \frac{d\rho}{dv} = -\frac{\rho}{c_s}. \tag{2.15}$$

Equation (2.15) gives a relation between the velocity v and the density ρ. In fact, the integration of Eq. (2.15) yields

$$v = \pm \int^{\rho} \frac{c_s}{\rho} d\rho. \tag{2.16}$$

Note that the sound velocity c_s is a function only of the density since it is isentropic motion.

- Substituting Eq. (2.15) into (2.14) yields

$$\frac{\partial v}{\partial t} + (v \pm c_s)\frac{\partial v}{\partial x} = 0. \tag{2.17}$$

It is clear that the density ρ satisfies a similar equation.

$$\frac{\partial \rho}{\partial t} + (v \pm c_s)\frac{\partial \rho}{\partial x} = 0. \tag{2.18}$$

By comparing these partial differential equations and the adiabatic condition (1.23), we can find that both the velocity v and the density ρ are constant along one of the lines $C_\pm$ in the x-t plane whose slopes are given by

$$\frac{dx}{dt} = v \pm c_s. \tag{2.19}$$

These lines are called **characteristics**. For simple waves, the corresponding characteristics is a straight line since the right-hand side of Eq. (2.19) is constant along the characteristic. If ρ and v are given as functions of x at time t_0, the distributions of ρ and v at any time can be obtained by finding the characteristic through each point (t_0, x) with the differential equation (2.19) (since ρ and v are constant along the characteristic). This solution method of partial differential equations is known as the **method of characteristics**.

Problem 7. Show that in an ideal gas, Eq. (2.16) is rewritten as

$$v = \pm\frac{2}{\gamma - 1}(c_s - c_{s,0}). \tag{2.20}$$

where $c_{s,0}$ is the sound velocity in a stationary gas and $\gamma > 1$. Also show that the slope of the characteristic (2.19) is equal to $(\gamma+1)v/2 \pm c_{s,0}$, and that the upper limit of the magnitude of the velocity $|v|$ is given by $2c_{s,0}/(\gamma-1)$ when $c_s < c_{s,0}$.

(b) Rarefaction waves

As a example of simple waves, let us consider a **rarefaction wave** (also called an **expansion fan**) in an ideal gas.

Problem Setup: There is a solid wall (piston) at $x = 0$ and a gaseous fluid at rest is filled in the space with $x > 0$. The initial density and sound velocity of gas are constants given by ρ_0 and $c_{s,0}$, respectively. At time $t = 0$, the solid wall (or the piston) begins to be pulled in the negative direction of x with a velocity of $-V$ ($V > 0$, see Figure 3, top). This motion of the piston generates a rarefaction wave in the gas. Since at $t = 0$ the gas at rest satisfies Eq. (2.16), the gas motion becomes a simple wave at subsequent instants. We further assume that the velocity of the piston is constant at $t > 0$.

Solution

- Even after the piston starts moving, the gas far enough away from the piston remains in the initial state ($v = 0$, $\rho = \rho_0$). On the other hand, the gas near the piston is pulled by the piston and has a negative velocity ($v < 0$). Thus, dv/dx is positive, which means an expansive motion. As a result, the density decreases near the piston, and then $d\rho/dx > 0$. Since the gradients of v and ρ have the same sign, a positive sign is chosen in Eqs. (2.15)-(2.20). Therefore, v and ρ are constant on the characteristic C_+, and C_+ is a straight line.

- The information that the piston has started moving is transmitted through the gas at $x > 0$ at the sound velocity $c_{s,0}$. The gas that has received this information moves to the negative direction of x. The characteristic C_+ through the origin of the x-t plane given by
$$\frac{x}{t} = c_{s,0} \tag{2.21}$$
is the boundary. The gas remains stationary on the far side of the boundary ($x/t > c_{s,0}$), and it has an expansive motion with $v < 0$ and $dv/dx > 0$ on the near side ($x/t < c_{s,0}$). The latter motion is the rarefaction wave.

- The velocity in the region where $x/t < c_{s,0}$ is obtained as follows. There exist other characteristics C_+ passing through the origin $x = t = 0$ with slopes different from Eq. (2.21). Because of the sudden start of the piston at $x = t = 0$, the nearby gas also experiences sudden and strong negative acceleration, which makes the velocity field discontinuous at $x = t = 0$. As a result, the slope of the characteristics C_+ through $x = t = 0$, which equals $v + c_s$, can have various values smaller than $c_{s,0}$. Then, using Eq. (2.20), we obtain for $x/t < c_{s,0}$,
$$\frac{x}{t} = v + c_s = \frac{\gamma+1}{2}v + c_{s,0}, \qquad \text{or} \qquad v = -\frac{2}{\gamma+1}\left(c_{s,0} - \frac{x}{t}\right). \tag{2.22}$$
This is the velocity field $v(x,t)$ in the rarefaction wave.

- There are two types of the gas flow in the region closer to the piston, depending on the speed of the piston.

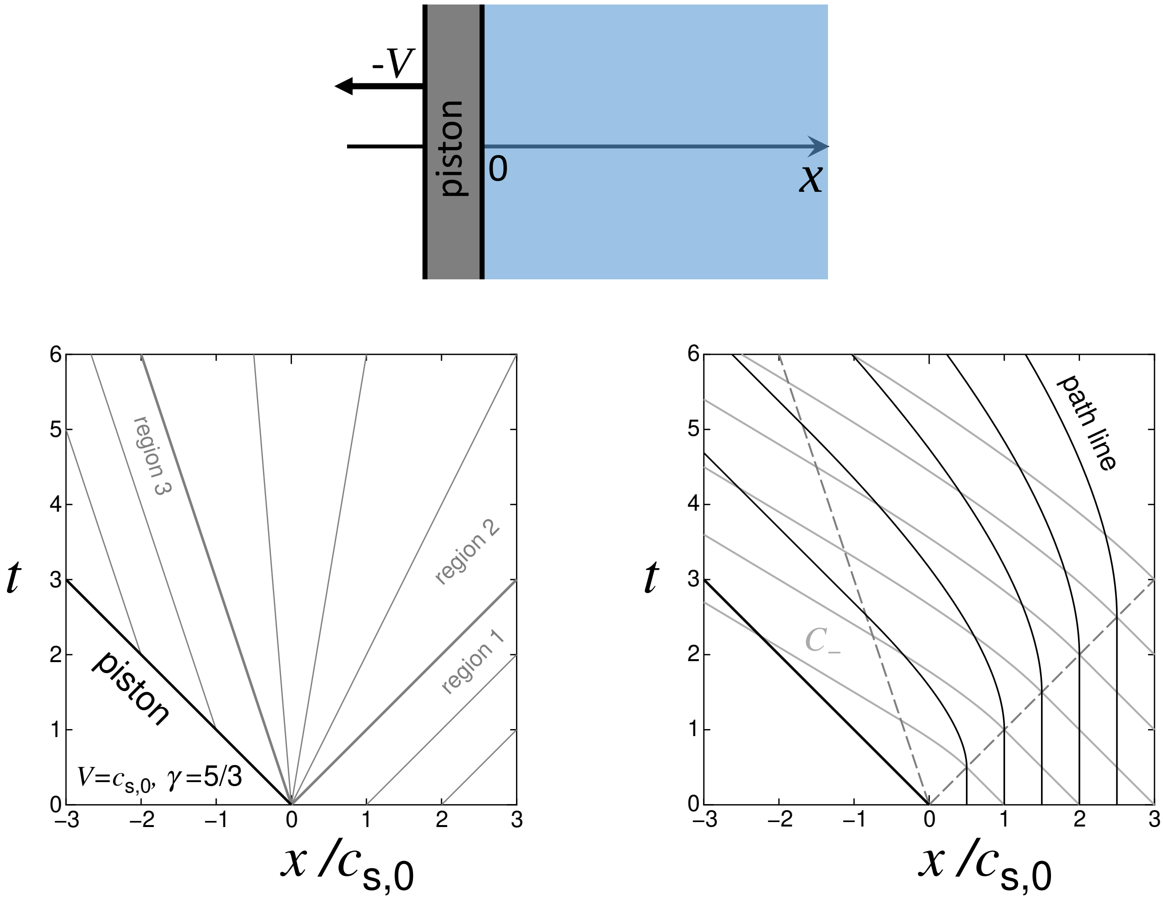


Figure 3: A rarefaction wave generated by the pulling of the piston. The schematic (top), the characteristics C_+ (bottom left), the characteristics C_-, and the path lines (bottom right) for the case of $V = c_{s,0}$ and $\gamma = 5/3$. In the bottom left panel, the characteristics C_+ and the motion of the piston are plotted by gray lines and the black line, respectively. The two thick gray lines show the boundaries of the rarefaction wave. In the bottom right panel, gray and black lines show the characteristics C_- and the path lines, respectively.

(i) If the speed of the piston, V, is lower than the upper limit $2c_{s,0}/(\gamma - 1)$, then the characteristic C_+ with $v = -V$ given by

$$\frac{x}{t} = -\frac{\gamma + 1}{2} V + c_{s,0} \tag{2.23}$$

is the other boundary of the rarefaction wave. In the region between this characteristic and the piston, the gas flows uniformly with the velocity of the piston and the gas density is constant. As an example, the bottom left panel of Figure 3 shows the characteristics C_+ in each region for the case of $V = c_{s,0}$ and $\gamma = 5/3$ while the bottom right panel shows the characteristics C_- and the path lines for the same case.

(ii) If $V > 2c_{s,0}/(\gamma - 1)$, the velocity of the piston exceeds the upper limit of the gas velocity, and thus, the gas cannot catch up with the piston and a vacuum region is formed. The boundary between the vacuum region and the rarefaction wave is given by the characteristic C_+ of $x/t = -2c_{s,0}/(\gamma - 1)$.

Problem 8. Write down the sound velocity and the density as functions of x and t in regions of the rarefaction wave and both sides, for the cases of (i) and (ii). Also check that the functions are continuous at the boundaries between these regions.

Problem 9. Solve the equation for the characteristics (2.19) in the rarefaction wave and obtain the equation of C_- in the form $x = f(t)$. Also solve the equation for the fluid path, $dx/dt = v$, and obtain the path lines in the same way. Let these two curves pass through the point $t = t_0$, $x = c_{s,0}t_0$.

(ASIDE) Let us take a closer look at the flow in Figure 3. The flow in Figure 3 consists of three regions: 1. the stationary gas, 2. the rarefaction wave, and 3. the uniform gas flow moving with the piston. The characteristics C_+ in Region 3 originate from the piston moving at the velocity -V, while C_+ in Region 1 come from the piston at rest at $t < 0$. The characteristics C_+ in Region 2 between these regions are considered to originate from the piston during its acceleration so that the velocity of the gas in the rarefaction wave takes $-V < v < 0$. Along each characteristic C_+, the information of the piston motion at that time propagate and determines the velocity and density of the gas on C_+.

The velocity field of the rarefaction wave (region 2) is independent of the "final velocity" of the piston, $-V$. This is because the gas in Region 2 only receives the information that the piston is accelerating, but not the information about the final velocity. On the other hand, the gas in Region 3 receives the information about the final velocity and moves at the same velocity as the piston.

The path lines and C_- are straight in Regions 1 and 3 since the flow is uniform. On the other hand, they become curved in the rarefaction wave. The gas gradually accelerates (to a negative velocity) and reaches $-V$ within the rarefaction wave. The angles between path lines and C_- gradually decrease in the downstream direction because the sound velocity (i.e., the velocity of information propagation) becomes slower.

(c) Riemann Invariants

Next we investigate more general waves in which v and ρ changes independently (but assuming isentropic motion). Adding or subtracting Eq. (2.10)$\times c_s/\rho$ to Eq. (2.11), we obtain

$$\frac{\partial v}{\partial t} \pm \frac{c_s}{\rho}\frac{\partial \rho}{\partial t} + (v \pm c_s)\left(\frac{\partial v}{\partial x} \pm \frac{c_s}{\rho}\frac{\partial \rho}{\partial x}\right) = 0. \tag{2.24}$$

We now introduce two variables defined by

$$J_\pm = v \pm \int \frac{c_s}{\rho} d\rho. \tag{2.25}$$

Using these, we can rewrite Eq. (2.24) in a simple form

$$\left[\frac{\partial}{\partial t} + (v \pm c_s)\frac{\partial}{\partial x}\right] J_\pm = 0. \tag{2.26}$$

These equations show that J_+ and J_- are constant along characteristics C_+ and C_-, respectively. The invariants $J_\pm$ are called **Riemann invariants**. The propagation of information along each characteristic make the Riemann invariants constant. In particular, for a polytropic ideal gas, the Riemann invariants are given by

$$J_\pm = v \pm \frac{2}{\gamma - 1} c_s. \tag{2.27}$$

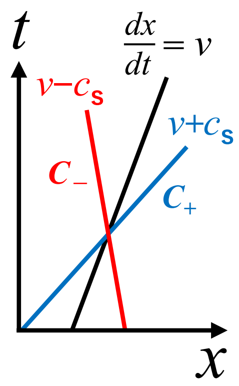
The partial differential equation (2.26) can be solved by finding the characteristics as in the case of the simple waves.

Problem 10. Derive Eq. (2.26).

Problem 11. Find the Riemann invariants $J_\pm$ for the rarefaction wave examined in (b) and each region on both sides of it.

(d) Generation of shock waves

Consider a wave packet propagating in the x direction with a density distribution as shown in the figure below. Assume that it is a simple wave and that the density and velocity are constant along characteristics C_+. Its propagation velocity is $v + c_s$. Since the sound velocity is greater where the density is higher, the denser part of the wave propagates faster than the other parts. As a result, the point of maximum density gradually overtakes the less dense leading part and eventually the density distribution $\rho(x)$ becomes multivalued ($t = 4\Delta$ in Figure 5). In reality, multivalued densities are not allowed, and a density discontinuity is created at the front of the wave. Then the velocity and pressure distributions also become discontinuous. This discontinuity is called a shock wave. Generally, a wave with a density distribution decreasing in the direction of propagation will create a shock wave if it travels a sufficient distance before attenuating. Shock waves are discussed in detail in the next section.



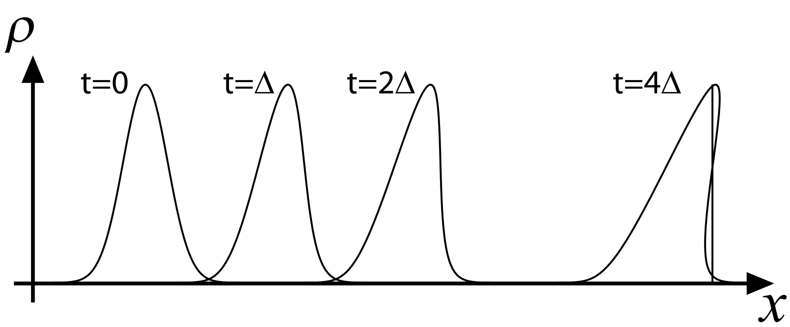


Figure 4: Characteristics $C_\pm$ and path line (left).
Figure 5: Propagation of a wavepacket and generation of a shock wave (right).

2.3 Stable discontinuities: shock waves and contact discontinuities

- We consider a surface at which quantities such as the density and the velocity are discontinuous. Generally, a surface of discontinuity moves, but here we consider a coordinate system in which the discontinuity surface is at rest. The normal direction of the discontinuity surface is taken as the x-axis, and the discontinuity surface is set at $x = 0$. We label the region with $x < 0$ as 1 and the positive x region as 2. The quantities in these regions are represented by subscripts 1 and 2.

- **Conditions for stable surfaces of discontinuity:** For a surface of discontinuity to exist stably, the mass flux, energy flux, and momentum flux (vector) across the surface must be continuous, which is required by conservations of the mass, energy, and momentum. Since the expressions of the flux densities are given by Eqs. (1.5), (1.19), and (1.37), the continuities of them yield the following equations.

$$\rho_1 v_1 = \rho_2 v_2 = j, \tag{2.28}$$

$$j[h_1 + \frac{1}{2}(v_1^2 + v_{1,y}^2 + v_{1,z}^2)] = j[h_2 + \frac{1}{2}(v_2^2 + v_{2,y}^2 + v_{2,z}^2)], \tag{2.29}$$

$$p_1 + \rho_1 v_1^2 = p_2 + \rho_2 v_2^2, \tag{2.30}$$

$$j v_{1,y} = j v_{2,y}, \tag{2.31}$$

$$j v_{1,z} = j v_{2,z}, \tag{2.32}$$

where v_1 and v_2 are x-components of the velocities in Region 1 and 2; and j is the x-component of the mass flux density. A discontinuity surface that does not satisfy these conditions instantly splits into multiple discontinuities or rarefaction waves, as will be seen in §§2.5.

- The stable surfaces of discontinuity that satisfy the above conditions are classified into two types. Discontinuities of the first type have non-zero j and are called **shock waves**. The second type with $j = 0$ is called a **tangential discontinuity**.

(a) Shock waves

- **Coordinate system moving with the shock wave**
Dividing Eqs. (2.31) and (2.32) by the non-zero mass flux density j, we find that v_y and v_z are continuous. In the following, we use the coordinate system where $v_{y,i} = v_{z,i} = 0$ to examine the shock wave. The direction of the x-axis is determined so that v_1, $v_2 > 0$ ($j > 0$). Then, Region 1 where $x < 0$ is upstream and is called the **pre-shock region**, and Region 2 where $x > 0$ is downstream and is called the **post-shock region**.

- **Coordinate-independent relations in shock waves**
  Equation (2.30) yields
$$j^2 = \frac{p_2 - p_1}{V_1 - V_2}, \tag{2.33}$$
  where $V_i = 1/\rho_i$ is the specific volume. Since the right hand side includes only thermodynamics quantities, it is independent of the coordinate system. From Eq. (2.33), we can see that either $p_2 > p_1$ and $V_1 > V_2$, or $p_2 < p_1$ and $V_1 < V_2$. As we will see later, the law of increasing entropy shows that the former case is always realized. That is, the pressure and density are higher at the post-shock region than at the pre-shock region. From Eqs. (2.28) and (2.33), we obtain
$$|\boldsymbol{v}_1 - \boldsymbol{v}_2|^2 = j^2(V_1 - V_2)^2 = (p_2 - p_1)(V_1 - V_2). \tag{2.34}$$
  Equation (2.29) is rewritten with Eq. (2.33) and $v_{y,i} = v_{z,i} = 0$ as
$$h_2 - h_1 = \frac{1}{2}j^2(V_1^2 - V_2^2) = \frac{1}{2}(V_1 + V_2)(p_2 - p_1). \tag{2.35}$$
  Furthermore, noting that $h = e + pV$, we also obtain
$$e_2 - e_1 = \frac{1}{2}(V_1 - V_2)(p_1 + p_2). \tag{2.36}$$

- **Shock adiabat**
  Generally, the enthalpy (or the internal energy) can be expressed by a function of the density and pressure using the equation of state. Substituting such a expression into Eq. (2.35) (or eq. [2.36]) we can obtain the pressure at the post-shock p_2 as a function of V_2 for given p_1 and V_1 at the pre-shock region. This relation in the p_2-V_2 plane is called a **shock adiabat** or a **Hugoniot curve**.

  If a thermodynamic quantity is given, then, we can obtain all other thermodynamic quantities using the shock adiabat and the equation of state. Furthermore, from Eqs. (2.33) and (2.34), we also obtain j, v_2, and v_1 (and the propagation velocity of shock against the gas in the pre-shock region is equals to $-v_1$). Thus the degree of freedom of a shock wave is one.

(b) Tangential discontinuity

In this case with $j = 0$, the pressure is continuous but v_y and v_z can be discontinuous. However, tangential discontinuities in which the tangential velocity is discontinuous are unstable due to a hydrodynamic instability, as will be seen in §§3.3. Therefore, stable tangential discontinuities are limited to cases where the the tangential velocity is continuous. Such tangential discontinuities are called **contact discontinuities**. In contact discontinuities, thermodynamic quantities other than pressure become discontinuous. The composition of fluid substances can also be discontinuous. The surface of the contact discontinuity moves with the fluid.

2.4 Shock waves in an ideal gas

- Let us derive **shock adiabats of an ideal gas**. For an ideal gas, the (specific) enthalpy is given by $h = \frac{\gamma}{\gamma-1}pV$. Substituting this into Eq. (2.35) and divide it by h_1, we obtain

$$\frac{p_2}{p_1}\frac{V_2}{V_1} - 1 = \frac{\gamma-1}{2\gamma}\left(\frac{V_2}{V_1}+1\right)\left(\frac{p_2}{p_1}-1\right). \tag{2.37}$$

This can be rewritten as

$$\frac{V_2}{V_1} = \frac{Ap_2/p_1+1}{p_2/p_1+A} \qquad \left(=\frac{\rho_1}{\rho_2}=\frac{v_2}{v_1}\right), \tag{2.38}$$

where the constant A is given by

$$A \equiv \frac{\gamma-1}{\gamma+1} \qquad (<1). \tag{2.39}$$

Equation (2.38) gives the relation between p_2 and V_2, i.e., the shock adiabats (see Figure 6). Even in the limit of strong shock ($p_2/p_1 \to \infty$), the specific volume decreases only to $V_2/V_1 = A$. We also obtain

$$\frac{V_2}{V_1} - 1 = -\frac{2}{\gamma+1}\frac{p_2/p_1-1}{p_2/p_1+A}. \tag{2.40}$$

Using Eq. (2.40), we obtain j, $v_1 - v_2$, and the Mach number in Region 1 M_1 as functions of p_2.

$$j^2 = \frac{\gamma+1}{2}\left(\frac{p_2}{p_1}+A\right)\frac{p_1}{V_1}, \tag{2.41}$$

$$|\boldsymbol{v}_1 - \boldsymbol{v}_2|^2 = \frac{2}{\gamma(\gamma+1)(p_2/p_1+A)}\left(\frac{p_2}{p_1}-1\right)^2 c_{s,1}^2, \tag{2.42}$$

$$M_1^2 \equiv \frac{v_1^2}{c_{s,1}^2} = \frac{\gamma+1}{2\gamma}\left(\frac{p_2}{p_1}+A\right). \tag{2.43}$$

The last equation shows $M_1 > 1$ for $p_2 > p_1$, indicating that the normal velocity is always supersonic in the pre-shock region and that shock waves propagate at a supersonic velocity against the pre-shock region.

Moreover, the post-shock quantities can be expressed as functions of M_1.

$$\frac{p_2}{p_1} = \frac{2\gamma}{\gamma+1}M_1^2 - A, \qquad \frac{V_2}{V_1} = \frac{2\gamma}{\gamma+1}M_1^{-2} + A. \tag{2.44}$$

- **Entropy increase due to shock waves**

From Eqs. (1.65) and (2.38), the change in the entropy is obtained as

$$s_2 - s_1 = c_V \ln\left[\frac{p_2}{p_1}\left(\frac{V_2}{V_1}\right)^\gamma\right] = c_V \ln\left[\frac{p_2}{p_1}\left(\frac{Ap_2/p_1+1}{p_2/p_1+A}\right)^\gamma\right]. \tag{2.45}$$

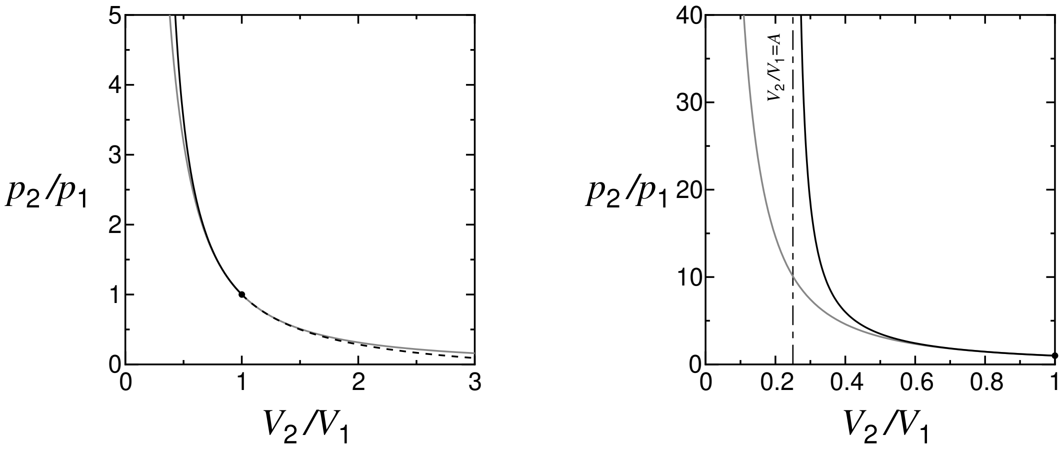


Figure 6: Shock adiabat (black line) and Poisson's adiabat (gray line) for ideal gas with $\gamma = 5/3$. For $V_2 < V_1$, the shock adiabat is above Poisson's adiabat due to the increase in the entropy. The left panel is the full view and the right is the magnified one for $V_2 < V_1$.

Differentiating this with p_2 gives

$$\frac{ds_2}{dp_2} = \frac{c_V A(p_2 - p_1)^2}{p_2(Ap_2 + p_1)(p_2 + Ap_1)} \tag{2.46}$$

and it is always positive. Since the entropy must increase with a shock wave, we find that $p_2 > p_1$. The entropy increase for a weak shock wave is given by

$$s_2 - s_1 = \frac{c_V A}{3(1+A)^2}\left(\frac{p_2 - p_1}{p_1}\right)^3 \qquad (\text{for } p_2 - p_1 \ll p_1)\,. \tag{2.47}$$

- **Generation and propagation of a shock wave by a piston**

  In §2.2, we saw that when the piston is pulled at a velocity of $-V$, a rarefaction wave is generated in the region where $x > 0$. Conversely, a shock wave is generated in the fluid at $x < 0$ pushed by the piston with velocity $-V$ (Figure 7). The gas is at rest in the pre-shock region (Region 1), where the shock wave has not yet reached. On the other hand, in the post-shock region (Region 2), the gas flows at the same velocity of the piston, $-V$. Thus we obtain $v_1 - v_2 = V$ as a function of the pressure ratio p_2/p_1 from Eq. (2.42). Also, the propagation velocity of the shock wave against the pre-shock gas is obtained from Eq. (2.43) since it equals $|v_1|$.

Problem 12. Derive Eqs. (2.38), (2.42), and (2.43).

Problem 13. Verify that $M_1 > 1$. Also show that the Mach number at post-shock, $M_2 (= v_2/c_{s,2})$, is less than unity.

2.5 Evolution of initial discontinuities: the Riemann problem

- **Riemann problem**

  Consider arbitrary initial discontinuities. If an initial discontinuity does not satisfy

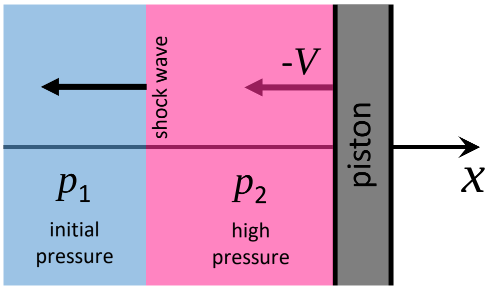


Figure 7: Shock wave by a piston.

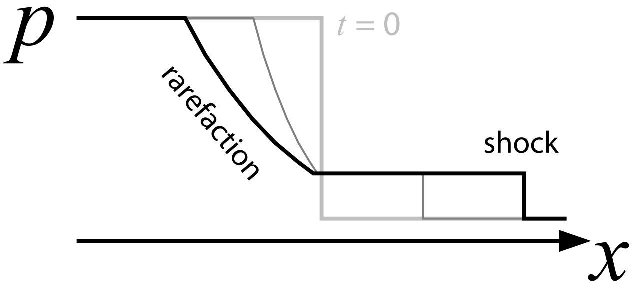


Figure 8: Pressure distribution in Sod's problem.

the conditions (2.28)-(2.30) for stable discontinuity surfaces, it generally splits into some stable discontinuities, such as rarefaction waves, shock waves, and contact discontinuities, and they move apart. The problem of solving the generation and propagation of the new discontinuities for an arbitrary initial discontinuity in an ideal gas is called the **Riemann problem**.

- **Riemann problem without the velocity jump**

  Let us consider a special case of the Riemann problem in which gases are at rest on both sides of the initial discontinuity, and there is no velocity jump. Let the surface of the initial discontinuity be at $x = 0$, and let the initial pressures and densities on both sides be denoted by p_1, ρ_1 for $x > 0$; and p_2, ρ_2 for $x < 0$, respectively[5]. We also assume that $p_2 > p_1$. We examine the time evolution of this discontinuity at $t > 0$.

  - This initial condition causes a flow in the x direction, from the high-pressure region $x < 0$ to the low-pressure region $x > 0$. As a result, a rarefaction wave with $dv/dx > 0$ appears in $x < 0$, and a shock wave propagates in $x > 0$ (Figure 2.4). The pressure and velocity are constant in the region between the rarefaction wave and the shock wave. Let these values be p_3 and v_3 (and set this region as 3).
  - Since this rarefaction wave has a negative $d\rho/dv$, the negative sign is chosen in Eqs. (2.15)-(2.20). Similar to Eqs. (2.21) and (2.23), both ends of the rarefaction wave is given by two characteristics C_-

$$\frac{x}{t} = -c_{s,2} \qquad \text{and} \qquad \frac{x}{t} = \frac{\gamma+1}{2}v_3 - c_{s,2}, \tag{2.48}$$

    respectively. Since the pre-shock gas is at rest in this case, the propagation velocity of the shock wave is given by "v_1". Also using Eq. (2.43), the position

[5]Note that the arrangement of Regions 1 and 2 is opposite to the case of §§**2.3**, which causes the inversion of signs in some equations.

of the shock wave, $x_{\rm sh}$, is given by

$$\frac{x_{\rm sh}}{t} = \sqrt{\frac{\gamma+1}{2\gamma}\left(\frac{p_3}{p_1}+A\right)}\, c_{s,1}. \qquad (2.49)$$

- In Region 3, there is a boundary between gases that were in contact at the surface of the initial discontinuity, and it is the surface of a contact discontinuity. Generally, the entropy (density and temperature) differs on both sides of a contact discontinuity. The position of the contact discontinuity surface is $x/t = v_3$.
- Determination of v_3 and p_3: From Eq. (2.20) for the rarefaction wave, we find that the velocity v_3 satisfies

$$v_3 = \frac{2}{\gamma-1}(c_{s,2}-c_{s,3}) = \frac{2c_{s,2}}{\gamma-1}\left[1-\left(\frac{p_3}{p_2}\right)^{\frac{\gamma-1}{2\gamma}}\right], \qquad (2.50)$$

  where the adiabatic relation between c_s and p is used in the last equality. Since v_3 is the velocity difference between the both sides of the shock wave, the following equation also holds for v_3.

$$v_3 = \sqrt{\frac{2}{\gamma(\gamma+1)(p_3/p_1+A)}}\left(\frac{p_3}{p_1}-1\right) c_{s,1}. \qquad (2.51)$$

  The equality of these two expressions for v_3 determines p_3 and v_3.
- The well-known **Sod's problem** is the above case where $p_2/p_1 = 10$, $\rho_2/\rho_1 = 8$. For $\gamma = 1.4$, we obtain $p_3/p_1 = 3.03130$ (noting that $c_{s,2}^2/c_{s,1}^2 = p_2/\rho_2/(p_1/\rho_1)$). The Sod's problem is useful for a test for compressible hydro-dynamical simulation codes.

Problem 14. Verify that the pressure ratio p_3/p_1 takes the above value in the Sod's problem, and find the ratio $v_3/c_{s,1}$. When ρ_3 and ρ_3' denote the densities on the larger-x side and the smaller-x side of the contact discontinuity, respectively, find the ratios ρ_3/ρ_1 and ρ_3'/ρ_1.

- **Riemann problem with a velocity jump**

  In general, the Riemann problem with a velocity jump can be classified into the following six cases, depending on the velocity jump between the two sides. In all cases, there is a contact discontinuity at the boundary between gases that were in contact at the initial discontinuity. First, we consider the cases of compressive velocity fields with $v_2 > v_1$. The arrangement of Regions 1 and 2 is the same as in the above.

  (i) When the initial velocity jump $|v_2 - v_1|$ is equal to that of the shock wave, Eq. (2.34), the initial discontinuity remains as a shock wave and propagates in Region 1 (i.e., on the low-pressure side).

(ii) When the velocity jump $|v_2 - v_1|$ is smaller than Eq. (2.34), it requires an additional expansive motion, and thus a shock wave and a rarefaction wave appear and propagate on the low-pressure side and the high-pressure side, respectively. Their arrangement is the qualitatively same as in the case without the velocity jump. By replacing the left-hand sides of Eqs. (2.50) and (2.51) by $|v_3 - v_2|$ and $|v_1 - v_3|$, respectively, we can obtain v_3 and p_3, as done above.

(iii) When the velocity jump $|v_2 - v_1|$ is larger than Eq. (2.34), the strong compression causes two shock waves to propagate to both sides. A high-pressure region is formed between the shock waves, and the pressure and the velocity in this region are constant. Conditions similar to Eq. (2.51) hold for the velocity jumps of the two shock waves, and we can obtain the pressure and the velocity in the region between the two shock waves. This case is caused by a collision of two gaseous objects.

The following are cases of the expansive velocity fields with $v_2 < v_1$.

(iv) When the expansive velocity jump $|v_1 - v_2|$ equals that of the rarefaction wave, Eq. (2.50), the initial discontinuity becomes a continuous rarefaction wave and propagates toward the high-pressure side.

(v) When the velocity jump $|v_1 - v_2|$ is smaller than Eq. (2.50), a shock wave and a rarefaction wave appear to propagate toward both sides as in case (ii) and the case without the velocity jump. We can obtain v_3 and p_3, as in case (ii).

(vi) When the velocity jump $|v_1 - v_2|$ is larger than Eq. (2.50), the strong expansion causes two rarefaction waves to propagate toward both sides. A low-pressure region with constant p_3 and v_3 is formed between the two rarefaction waves, and p_3 and v_3 can be obtained from conditions similar to Eq. (2.50) for the velocity differences in each rarefaction wave.

Problem 15. Find the pressure distribution for the Riemann problem where only a velocity jump exists initially. Consider both compressive and expansive cases, setting $v_1 = -v_2$.

2.6 Transition to supersonic speed in a steady flow

- Consider a steady flow of an isentropic gas. Gravity is assumed to be ineffective. Since Bernoulli's equation (1.31) holds for an isentropic steady flow, the velocity satisfies

$$v^2 = 2(h_0 - h) = \frac{2\gamma}{\gamma - 1}\left(\frac{p_0}{\rho_0} - \frac{p}{\rho}\right) = \frac{2}{\gamma - 1}(c_{s,0}^2 - c_s^2), \tag{2.52}$$

where the thermodynamic quantities with the subscript 0 represent the values at $v = 0$. This equation indicates that the gas accelerates as the pressure, density, and sound velocity decrease. As the acceleration continues, the velocity eventually exceeds the sound velocity, and the flow becomes a supersonic. The transition flow from subsonic to supersonic is called the transonic flow. The point where the Mach number $M = v/c_s = 1$ is called the critical point or the sonic point. The critical velocity, $c_{s,*}$, is the sound velocity at this point, which is obtained as

$$c_{s,*} = \sqrt{\frac{2}{\gamma+1}}\, c_{s,0}. \tag{2.53}$$

Using Eq. (2.52), each thermodynamic quantity is obtained as a function of the velocity.

$$\left(\frac{c_s}{c_{s,0}}\right)^2 = \left(\frac{\rho}{\rho_0}\right)^{\gamma-1} = 1 - \frac{\gamma-1}{2}\left(\frac{v}{c_{s,0}}\right)^2 = 1 - \frac{\gamma-1}{\gamma+1}\left(\frac{v}{c_{s,*}}\right)^2. \tag{2.54}$$

- **Mass flux density in the steady transonic flow**

  Using Eq. (2.54), the magnitude of the mass flux density, $j = \rho v$, can be obtained as a function of v (Figure 9, left), showing that j reaches its maximum value at the critical point. This can be shown generally from the steady-state Euler's equation $\boldsymbol{v} \cdot \mathbf{grad}\, \boldsymbol{v} = -(1/\rho)\mathbf{grad}\, p$. Since this equation gives $dp/dv = -\rho v$ along a streamline, we obtain

$$\frac{d\rho}{dv} = \frac{dp}{dv} \Big/ \left(\frac{dp}{d\rho}\right)_s = -\frac{\rho v}{c_s^2}. \tag{2.55}$$

  Thus, dj/dv is given by

$$\frac{dj}{dv} = \rho + v\frac{d\rho}{dv} = \rho\left[1 - \left(\frac{v}{c_s}\right)^2\right]. \tag{2.56}$$

  This shows that j increases with v at subsonic speeds, and decreases at supersonic speeds. Therefore, j reaches its maximum value $j_* = \rho_* c_{s,*}$ at the critical point. (The subscription $*$ indicates quantities at the critical point.)

- **Flow in a de Laval nozzle**

  - To accelerate the gas to supersonic speeds, j must be varied as shown in the left panel of Figure 9. Let us consider how this can be achieved.
  - Consider a steady flow in a tube with a cross section S that varies in the direction of the x-axis along the tube. Also, assume that the flow in the tube is one-dimensional and only depends on x. For a steady flow in a tube, the mass flux through each cross section of the tube is constant, i.e.,

$$S\, j = \text{const.} \tag{2.57}$$

    Thus, we obtain

$$\frac{1}{S}\frac{dS}{dx} = -\frac{1}{j}\frac{dj}{dx} = \left[\left(\frac{v}{c_s}\right)^2 - 1\right]\frac{1}{v}\frac{dv}{dx}, \tag{2.58}$$

    where Eq. (2.56) is used in the second equality. This equation determines the one-dimensional velocity field in the tube.

- This equation shows that j and v are constant in a tube with a constant cross section S. For a tube that tapers ($dS/dx < 0$), j increases and the flow accelerates in the subsonic range but does not reach supersonic speeds. Therefore, to create a flow exceeding the sound velocity, it is necessary to use a tube with a minimum cross-section $S_{\min}$ in the middle of the tube, as shown in the right panel of Figure 9. If the velocity becomes equal to the sound velocity at $S = S_{\min}$, j decreases beyond that point, making it possible to continue accelerating even at supersonic speeds. A tube with this shape is called de Laval nozzle.
- A supersonic gas ejection mechanism using de Laval nozzle is applied to the rocket jet engine. The flow that exceeds the critical point in de Laval nozzles is also useful for understanding the acceleration of stellar winds and other supersonic flows in the universe.

Problem 16. From Eq. (2.58), we see that dv/dx is proportional to $\sqrt{d^2S/dx^2}$. Find the proportional coefficient. (Hint) Use L'Hôpital's rule for 0/0. Note also that c_s depends on x.

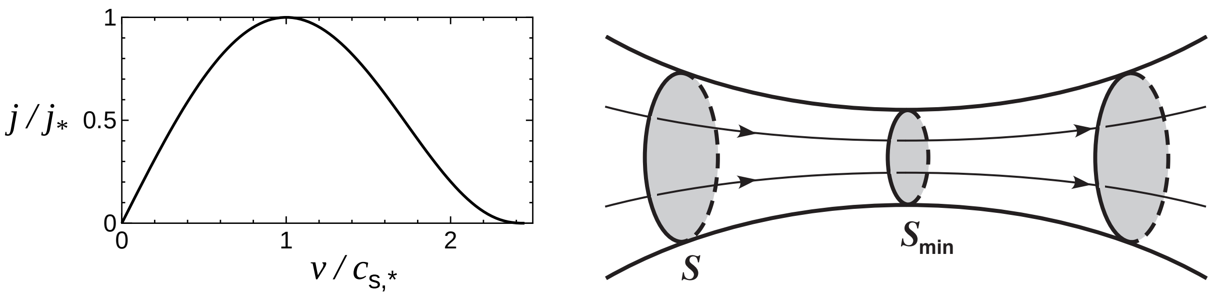


Figure 9: The j-v relation for the gas with $\gamma = 1.4$ (left) and a flow in de Laval nozzle (right).

3 Hydrodynamic Stability

3.1 Hydrostatic equilibrium

- Let us first describe the structure of a fluid at rest in a gravitational field. For a fluid at rest ($v = 0$), Euler's equation including the gravitational force can be written as

$$-\frac{1}{\rho}\,\mathbf{grad}\,p - \mathbf{grad}\,\phi = 0, \tag{3.1}$$

  where ϕ is the gravitational potential, which is governed by Poisson's equation

$$\triangle\phi = 4\pi G\rho. \tag{3.2}$$

  Equation (3.1) is called the hydrostatic equation.

- In the case of a uniform gravitational field, the direction of the z-axis is usually chosen to be opposite to the gravitational force. Then the gravitational force is written as $-\mathbf{grad}\,\phi = -g\,\boldsymbol{e}_z$, and the hydrostatic equation is given by

$$\frac{dp}{dz} = -g\rho. \tag{3.3}$$

- A non-rotating hydrostatic body will have a spherical structure, and its density distribution will be spherically symmetric. The gravitational force of such a body is given by

$$-\mathbf{grad}\,\phi = -\frac{GM(r)\boldsymbol{r}}{r^3}, \tag{3.4}$$

  where $M(r)$ is the mass contained in a sphere of the radius r given by

$$M(r) = \int_0^r 4\pi\rho r^2 dr. \tag{3.5}$$

  Then the hydrostatic equation for a spherically symmetric self-gravitating body is written as

$$\frac{dp}{dr} = -\frac{GM(r)\rho}{r^2}. \tag{3.6}$$

  This equation is used to study the structure of a spherically symmetric star.

- We have assumed a perfectly static fluid above. However, the hydrostatic equation is also valid if the convective motion is sufficiently slow compared to the sound velocity.

- To solve the problem of the hydrostatic equilibrium, we also need the energy equation (or the adiabatic condition) together with the hydrostatic equation. The energy equation generally describes the energy transfer due to the radiation and the convective motion.

- We studied the component of the hydrostatic equation parallel to the gravitational force. The horizontal components perpendicular the gravitational force require the pressure and the density to be constant on the gravitational equi-potential surface. This also makes the equipotential surface coincide with the isothermal surface. This is an important property of the hydrostatic equilibrium.

Problem 17. Derive the hydrostatic equation (3.4) for a spherically symmetric self-gravitating body from Poisson's equation (3.2).

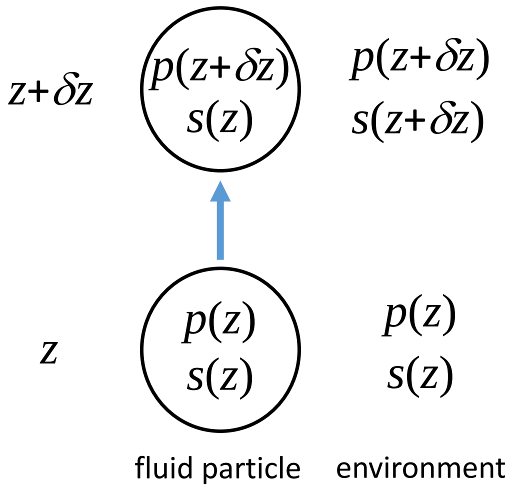


Figure 10: Thermodynamic states of an upwardly displaced particle and its environment

3.2 Stable condition against convection

- Mechanical equilibrium in which forces are balanced, such as hydrostatic equilibrium or steady flows, is not necessarily stable. In general, it is stable only if certain conditions are satisfied. Let us consider the conditions that determine whether an equilibrium state is stable or not. When a hydrostatic equilibrium state is unstable, flow occurs spontaneously, for example, convection occurs. In this section, we derive the condition for the absence of convection.

- In a hydrostatic structure in a uniform gravitational field, the pressure p and entropy s are given as functions of z from the hydrostatic equation and the energy equation. Consider a fluid particle at height z with a specific volume $V(p(z), s(z))$. Suppose that the fluid particle moves adiabatically upwards by a small interval δz. The specific volume after the displacement is given by $V(p(z+\delta z), s(z))$ because the displacement is adiabatic and the pressure quickly equilibrates with the surroundings. If the upwardly displaced fluid particle has a higher density than the surrounding fluid at height $z+\delta z$, an additional gravitational force due to the excess density pushes it back to the original height z. In this case, convection does not occur and we can say that the hydrostatic equilibrium state is stable. Since a higher density means a smaller specific volume, the condition for the absence of convection is written as

$$V(p(z+\delta z), s(z+\delta z)) - V(p(z+\delta z), s(z)) > 0. \tag{3.7}$$

The first term on the left-hand side of the above inequality is the specific volume of the surrounding fluid. Expanding this first term by the difference $s(z+\delta z) - s(z) = (ds/dz)\delta z$, the condition becomes

$$\left(\frac{\partial V}{\partial s}\right)_p \frac{ds}{dz} > 0 \tag{3.8}$$

We can also obtain the same condition for a negative displacement δz. The thermodynamic formula gives

$$\left(\frac{\partial V}{\partial s}\right)_p = \frac{T}{c_p}\left(\frac{\partial V}{\partial T}\right)_p = \frac{\gamma - 1}{\gamma}\frac{m}{k_B}V \tag{3.9}$$

The second equality holds for ideal gases. For ideal gases, $(\partial V/\partial s)_p$ is always positive. Many other substances also expand when heated and have positive $(\partial V/\partial T)_p$ and $(\partial V/\partial s)_p$. Therefore, for most substances, the stable condition against convection becomes[6]

$$\frac{ds}{dz} > 0. \tag{3.10}$$

- The condition (3.10) can be expressed with the temperature gradient. Using Maxwell's relations and other thermodynamic formulas, the condition (3.10) is rewritten as

$$\frac{ds}{dz} = \left(\frac{\partial s}{\partial T}\right)_p \frac{dT}{dz} + \left(\frac{\partial s}{\partial p}\right)_T \frac{dp}{dz} = \frac{c_p}{T}\frac{dT}{dz} - \left(\frac{\partial V}{\partial T}\right)_p \frac{dp}{dz} \tag{3.11}$$

Furthermore, using the hydrostatic equation (3.3), we obtain the stable condition as

$$\frac{dT}{dz} > -\frac{gT}{c_p V}\left(\frac{\partial V}{\partial T}\right)_p \tag{3.12}$$

This shows that a positive temperature gradient dT/dz always results in a stable hydrostatic structure. Even a negative temperature gradient results in a stable structure if the absolute value of dT/dz is less than $(gT/c_pV)(\partial V/\partial T)_p$. This threshold for $|dT/dz|$ is called the adiabatic temperature gradient. For ideal gases, the adiabatic temperature gradient is given by g/c_p.

- Let's also derive the restoring force against the vertical displacement of a fluid particle mentioned above. The additional gravitational force due to a change in density is the buoyancy. Using the density difference $\delta\rho$ with the surroundings, the buoyancy per unit volume is given by

$$f_b = -g\delta\rho. \tag{3.13}$$

[6] Note that water has a negative $(\partial V/\partial T)_p$ at $T = 0 - 4$°C.

Furthermore, $\delta\rho$ is equal to $1/V(p(z+\delta z), s(z)) - 1/V(p(z+\delta z), s(z+\delta z))$, and transformations of $\delta\rho$ similar to Eqs. (3.8) and (3.9) yield

$$f_b = -\frac{gT}{c_p V^2}\left(\frac{\partial V}{\partial T}\right)_p \frac{ds}{dz}\delta z \tag{3.14}$$

This restoring force causes the fluid particles to oscillate. Its frequency is obtained from Eq. (3.14) as

$$\omega^2 = -\frac{f_b}{\rho\delta z} = \frac{gT}{c_p V}\left(\frac{\partial V}{\partial T}\right)_p \frac{ds}{dz} \tag{3.15}$$

This is called the Brandt-Vaisala frequency. In a stable hydrostatic equilibrium, the waves oscillating due to this buoyancy are called internal gravity waves.

3.3 Instability of tangential discontinuities

- Consider an incompressible fluid with a tangential discontinuity. The tangential discontinuity lies in the horizontal plane at z = 0 in a uniform gravitational field. The fluid below the discontinuity is denoted by 1 and has a density of ρ_1, and the fluid above the discontinuity is denoted by 2 and has a density of ρ_2. Due to the discontinuity of the tangential velocity, fluid 2 "slides" on fluid 1.

  Suppose that the fluid is in a state that slightly deviates from a hydrostatic equilibrium, i.e., a state in which small perturbations are added to the vertical hydrostatic equilibrium. We derive the governing equations of the perturbations to study their time evolution. If the perturbations grow infinitely, the equilibrium state is unstable, and if they remain small, the equilibrium state is stable.

  The pressure and velocity in a state where perturbations are added are written as

$$p = P + \delta p, \qquad v_x = U + u, \qquad v_z = w, \tag{3.16}$$

  where δp, u, and w are the perturbations. For simplicity, these perturbations are assumed to be independent of y and the y-component of the velocity perturbation is set to be zero. The unperturbed equilibrium state has an x-component of the velocity, U. It is equal to a constant U_1 for $z < 0$ and is a constant U_2 for $z > 0$.

- Let us derive the equations for the perturbations. To do this, we substitute Eq. (3.16) into the hydrodynamic equations, leaving only the first-order terms of the perturbations. Generally, an analysis with the equations of the first-order terms of the perturbations is called a linear stability analysis. From the equation of continuity for incompressible fluids, div $\boldsymbol{v} = 0$, we obtain

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0 \tag{3.17}$$

  The x- and z-components of Euler's equation yield

$$\rho\left(\frac{\partial u}{\partial t} + U\frac{\partial u}{\partial x} + w\frac{dU}{dz}\right) = -\frac{\partial \delta p}{\partial x}, \tag{3.18}$$

$$\rho\left(\frac{\partial w}{\partial t}+U\frac{\partial w}{\partial x}\right)=-\frac{\partial \delta p}{\partial z}+f_b. \tag{3.19}$$

The last term, dU/dz, on the left side of Eq. (3.18) is zero except for $z=0$. In Eq. (3.19), f_b is the buoyancy. The expression for the buoyancy will be described later.

In addition to the above equations, we need the equation describing the boundary separating fluids 1 and 2 (i.e., the discontinuous surface), which originally lies at z = 0. Since the location of the boundary $z=\zeta$ shifts due to the vertical motion of the fluid particles near the boundary, the equation for ζ is give by

$$\frac{D\zeta}{Dt}\equiv\frac{\partial\zeta}{\partial t}+U\frac{\partial\zeta}{\partial x}=w(z=0). \tag{3.20}$$

The expression of the buoyancy is derived as follows. The buoyancy is given by $f_b=-g\delta\rho$. Since incompressible fluids are considered, there are no density changes inside fluids 1 and 2 and the buoyancy vanishes there. However, near the boundary, a shift of the boundary can cause the density to change from ρ_1 to ρ_2 or vice versa, and buoyancy appears. In the case of a positive ζ, the buoyancy is given by

$$f_b=\begin{cases}-g(\rho_1-\rho_2) & (0<z<\zeta),\\ 0 & (\text{otherwise}),\end{cases} \tag{3.21}$$

and for a negative ζ,

$$f_b=\begin{cases}-g(\rho_2-\rho_1) & (\zeta<z<0),\\ 0 & (\text{otherwise}).\end{cases} \tag{3.22}$$

The obtained buoyancy is not small but can be regarded as a perturbation because it works only in a narrow region, as we will see later. Eqs. (3.17)-(3.22) determine the linear perturbations around the tangential discontinuity.

- The above equations have a solution in which each perturbation depends exponentially on t and x as

$$w(t,x,z)=w'(z)\exp[i(kx-\omega t)]. \tag{3.23}$$

Other perturbations δp, u, and ζ are also written in the same form. In Eq. (3.23), the wave number k is a positive real number, and the (angular) frequency ω is generally a complex number[7]. If ω is a complex number and its imaginary part is positive, then, the perturbation w increases exponentially with time and the unperturbed state is unstable. Therefore, we can examine the stability, by checking whether ω is such a complex number.

[7] Note that w' is also complex. Exactly speaking, we should consider only the real part of the right-hand side of Eq. (3.23). However, if we also include the imaginary part, we can simplify the subsequent calculations.

- Substituting the expression of Eq. (3.23) for each perturbation into Eqs. (3.17), (3.18), and (3.20), and omitting $\exp[i(kx - \omega t)]$ in all terms, we obtain

$$iku' + \frac{dw'}{dz} = 0, \tag{3.24}$$

$$-i\rho(\omega - kU)u' + \rho w' \frac{dU}{dz} = -ik\,\delta p', \tag{3.25}$$

$$-i(\omega - kU)\zeta' = w'(z = 0). \tag{3.26}$$

Equation (3.19) is treated differently near the boundary where the buoyancy works, and the rest of the region. At $|z| > |\zeta'|$, the buoyancy vanishes (see Eqs [3.21] and [3.22]) and we obtain

$$-\,i\rho(\omega - kU)w' = -\frac{d\,\delta p'}{dz} \qquad (\,z > |\zeta'| \text{ or } z < |\zeta'|\,). \tag{3.27}$$

On the other hand, for the interval of $-|\zeta'| < z < |\zeta'|$, we integrate Eq. (3.19) over this interval. Each term on the left side becomes a second-order term for small perturbations due to the integration over the narrow interval $2|\zeta|$ and can be ignored. Integrating the right-hand side, we obtain

$$-\,[\,\delta p'(z = |\zeta'|) - \delta p'(z = -|\zeta'|)\,] - g(\rho_1 - \rho_2)\zeta' = 0. \tag{3.28}$$

This equation indicates that the buoyancy makes the pressure perturbation discontinuous at the boundary. Equations (3.26) and (3.28) are the boundary conditions for the perturbations at $z \simeq 0$.

- Solve Eqs. (3.24), (3.25), and (3.27) for the regions of the fluids 1 and 2. By eliminating u' from Eqs. (3.24) and (3.25) in each region and noting that $dU/dz = 0$ for $z \neq 0$, we obtain

$$\rho\,(\omega - kU)\,\frac{dw'}{dz} = -ik^2\,\delta p'. \tag{3.29}$$

Using the z-derivative of this equation, we also eliminate δp in Eq. (3.27) and obtain

$$\frac{d^2 w'}{dz^2} - k^2 w' = 0. \tag{3.30}$$

Assuming that the perturbations are not divergent as $z \to \pm\infty$, we obtain the solution in each region as

$$\begin{cases} w'_1 = A_1 e^{kz} & (z < 0), \\ w'_2 = A_2 e^{-kz} & (z > 0), \end{cases} \tag{3.31}$$

where $A_i = w'_i(z = 0)$. Generally, $w'(z)$ is discontinuous at the boundary.

- We impose the boundary conditions at $z = 0$ in the solution. Using Eqs. (3.29) and(3.31), Eq. (3.28) becomes

$$i\,[\rho_2(\omega - kU_2)w'_2(z = 0) + \rho_1(\omega - kU_1)w'_1(z = 0)] - kg(\rho_1 - \rho_2)\zeta' = 0 \tag{3.32}$$

and further eliminating w_i' using Eq. (3.26), we finally obtain

$$\rho_2(\omega - kU_2)^2 + \rho_1(\omega - kU_1)^2 - kg(\rho_1 - \rho_2) = 0. \tag{3.33}$$

This gives the relation between the frequency ω and the wave number k and is called the dispersion relation. The dispersion relation determines whether ω is complex or not.

- **Rayleigh-Taylor instability**

We first consider the case of $U_1 = U_2 = 0$, where only the density is discontinuous at the boundary. The instability of the density discontinuity is called the Rayleigh-Taylor instability In this case the dispersion relation becomes

$$\omega^2 = kg\,\frac{\rho_1 - \rho_2}{\rho_1 + \rho_2}. \tag{3.34}$$

When $\rho_2 > \rho_1$, the right-hand side of Eq. (3.34) becomes negative, and the frequency ω becomes a purely imaginary number, such as $\omega = \pm i\alpha$, where $\alpha = \sqrt{kg(\rho_2 - \rho_1)/(\rho_1 + \rho_2)}$ is a positive real number. Since each perturbation is proportional to $\exp(-i\omega t)$, a perturbation with a mode of $\omega = +i\alpha$ grows exponentially as $\exp(\alpha t)$. Therefore, we find that the state where $\rho_2 > \rho_1$ is unstable (although it was obvious). The growth rate α of this mode is proportional to $k^{1/2}$. This means that the mode with the shorter wavelength grows faster. Conversely, when $\rho_2 < \rho_1$, since the frequency ω is real, the perturbation only oscillates and does not grow, and this state is stable.

The waves that propagate this stable discontinuous surface are called the surface gravity waves, and they propagate along the interface at a speed of ω/k. Although we have considered the case of a constant gravitational field, the Rayleigh-Taylor instability also occurs when the fluid undergoes accelerated motion (or decelerated) and is subjected to the inertial force instead of the gravity. For example, when the outer shell of a massive star blown away by its supernova explosion interacts with interstellar gas and decelerates, the inertial force, which has the opposite direction to the stellar gravity, results in the Rayleigh-Taylor instability.

- **Kelvin-Helmholtz instability**

We next consider the case with a non-zero velocity jump. This instability at the discontinuity of the tangential velocity is called the Kelvin-Helmholtz instability. Equation (3.33) is a quadratic equation for ω. The solution is

$$\omega = \frac{k}{\rho_1 + \rho_2}\left[\rho_1 U_1 + \rho_2 U_2 \pm \sqrt{g(\rho_1^2 - \rho_2^2)/k - \rho_1\rho_2(U_1 - U_2)^2}\,\right]. \tag{3.35}$$

Therefore, for the wave number k greater than $k_{\rm min}$ defined by

$$k_{\rm min} = \frac{g(\rho_1^2 - \rho_2^2)}{\rho_1\rho_2(U_1 - U_2)^2} \tag{3.36}$$

the frequency ω is complex. Then the modes in which the imaginary part of ω is positive grow exponentially. Since such modes always exist under realistic conditions, it is concluded that the discontinuity of the tangential velocity is always unstable.

3.4 Turbulence

(a) Onset of turbulence

The flow of a viscous fluid depends on the Reynolds number $R_e = UL/\nu$, where U and L are the characteristic velocity and length of the flow, and ν is the kinematic viscosity. When the Reynolds number is sufficiently small, the viscosity acts effectively, the flow becomes steady, and the velocity changes smoothly with position. When the Reynolds number exceeds a threshold value, the original steady flow becomes unstable, and perturbations grow. If the Reynolds number is slightly above the threshold, the viscosity regulates the instability, resulting in another steady flow or an unsteady but regular flow with periodicity. On the other hand, as the Reynolds number increases, the regular and periodic motion becomes a superposition of several modes of frequencies and wavenumbers, resulting in a complex flow. When the Reynolds number becomes sufficiently large, the flow becomes very complex and unpredictable. This type of flow is called turbulence.

Turbulence can be thought of as the superposition of a huge number of eddies of different sizes. Turbulence is characterized by its variability irregularity, and unpredictability. For this reason, it is impossible and physically meaningless to describe turbulence accurately at each position. However, due to the complexity and enormous number of degrees of freedom, statistical methods are valid for describing turbulence. (This is the same as thermo-statistical mechanics being valid for macroscopic objects.)

(b) Statistical properties of turbulence

Consider a homogeneous and isotropic turbulent flow. The velocity field of the flow is divided into the mean velocity and the deviation from it. This deviation is the fluctuating part of the velocity field and is characteristic of turbulence. The average amplitude of the fluctuating part is denoted by Δv. The fluctuating part is a superposition of components of different wavelengths, and the largest wavelength in the components is denoted by L. It can be said that the size of the largest eddies (or vortices) in the turbulent flow is L and that the velocity of the largest eddies is Δv. These two quantities characterize the turbulence. The Reynolds number of turbulence given by $Re = \Delta v L/\nu$ is enormous The amplitudes of the pressure and density fluctuations are also determined by the largest vortices and estimated as $\Delta p \sim \rho(\Delta v)^2$ and $\Delta \rho \sim \rho(\Delta v/c_s)^2$, respectively.

For smaller vortices, their velocity amplitudes decrease with their size l, and their kinetic energy also decreases. On the other hand, the velocity gradient $dv/dx \sim v_l/l$ increases with decreasing size l. Turbulence consists of all vortices of different scales, and the properties of these vortices can be understood by the concept of the turbulent cascade.

- **Turbulent cascade**

  The complex structure of turbulence is formed by a mechanism called the turbulent cascade. The turbulent cascade consists of three stages.

  The first stage is the excitation of the largest vortices by external action. External action is caused by various effects, such as gravity, magnetic fields, and collisions. The nature of the largest vortices is determined by the external action. Let the energy injection rate per unit fluid mass into the largest vortices due to external action be ϵ [J/kg/sec].

  The second stage is the process of the vortex breakup, resulting in the creation of smaller vortices. The decay time (lifetime) of a vortex is approximately its period and estimated as l/v_l.

  The created small vortices eventually break up, generating even smaller vortices. By repeating the creation and breakup, vortices have a wide size range, from the largest vortices to small vortices. This phenomenon is called a turbulence cascade because small vortices are created in a chain. Since the kinetic energy is transferred from large vortices to smaller vortices in a turbulent cascade, it is also called an energy cascade.

  The third and final stage of the turbulent cascade is the viscous energy dissipation. Viscous energy dissipation is determined by the velocity gradient and occurs primarily in the smallest vortices.

- **Velocity distribution in a turbulent cascade**

  Consider the energy transfer in the turbulent cascade. As mentioned above, the kinetic energy is first injected into the largest vortices at ϵ [J/kg/sec]. From them, as the vortices break up sequentially, the kinetic energy is gradually transferred to smaller vortices and finally dissipated due to viscosity in the smallest vortices.

  The energy transfer rate [J/kg/sec] of each size range of vortices to the smaller size is estimated by (the kinetic energy per unit mass at each size) / (the decay time), that is

  $$v_l^2/(l/v_l) \sim v_l^3/l. \tag{3.37}$$

  Since the energy transfer from large size to small proceeds steadily on average, the energy transfer rate at each size to the smaller size is independent of the size and equal to ϵ. Therefore, we obtain

  $$\epsilon \sim v_l^3/l, \quad \text{or} \quad v_l \sim (\epsilon l)^{1/3}. \tag{3.38}$$

  Since the former equation is written as $\epsilon \sim (\Delta v)^3/L$ for the largest vortices, the velocity at each size is obtained as

  $$v_l \sim \Delta v (l/L)^{1/3}. \tag{3.39}$$

  The frequency of each vortex is estimated as $\omega \sim v_l/l$. Using this relation, we can also write the vortex velocity as a function of the frequency ω. Since this relation gives $v_\omega = v_l \sim (\epsilon v_l/\omega)^{1/3}$, we obtain

  $$v_\omega \sim (\epsilon/\omega)^{1/2}. \tag{3.40}$$

- **Size of the smallest vortices**
  For the smallest vortices, the energy transfer rate is equal to the energy dissipation rate due to viscosity. The latter is estimated to be $\nu(v_l/l)^2$[8]. Since both are equal at the smallest size l_0, its Reynolds number becomes $Re(l_0) = v_{l_0} l_0/\nu \sim 1$. Substituting Eqs. (3.38) and (3.39) into this relation for $Re(l_0)$, we obtain the size and velocity of the smallest vortices as

$$l_0 \sim (\nu^3/\epsilon)^{1/4} \sim Re(L)^{-3/4} L, \qquad v_{l_0} \sim Re(L)^{-1/4} \Delta v, \tag{3.41}$$

  where $Re(L) = \Delta v L/\nu \gg 1$. The size range of $l_0 \ll l \ll L$ is called the inertial range since there is almost no energy injection from the external or no energy dissipation due to viscosity in this range.

- **Energy distribution in a turbulent cascade**
  Find the energy distribution for size $E(l)$. The vortex energy per unit mass for a size l satisfies $E(l)\Delta l \sim E(l) l \sim v_l^2$, where the size width Δl is set to be comparable to l. Then, we obtain

$$E(l) \sim (\epsilon^2/l)^{1/3} \sim [(\Delta v)^2/L]\,(l/L)^{-1/3} \tag{3.42}$$

  Also, the distribution $E(k)$ for a wavenumber $k(= 1/l)$ is obtained as

$$E(k) \sim v_l^2/k \sim \epsilon^{2/3} k^{-5/3} \sim (\Delta v)^2 L\,(Lk)^{-5/3} \tag{3.43}$$

  The energy distribution $E(\omega)$ for the frequency ω is given by

$$E(\omega) \sim \epsilon/\omega^2. \tag{3.44}$$

  Problem 18. Find the dimensions of the energy distribution functions, $E(l)$, $E(k)$, and $E(\omega)$.

- **Diffusion process in turbulence**
  In a turbulent flow, the rotational motion of the many vortices strongly enhances the transport of momentum and energy. This transport is diffusive because the turbulence is irregular. The strong diffusion in the turbulent flow can be thought of as if the turbulent flow had a large viscosity $\nu_{\rm turb}$, which is called the eddy viscosity or turbulent viscosity. Diffusion due to turbulent motion is mainly caused by the largest vortices, and the turbulent viscosity is given by

$$\nu_{\rm turb} \sim \Delta v L. \tag{3.45}$$

  If we hypothetically consider a fluid with a viscosity of $\nu_{\rm turb}$, $Re(L)$ would be unity, and the viscous dissipation rate would be equal to the energy injection rate ϵ at the largest vortices. This means that the energy dissipation occurs in the largest vortices in the hypothetical fluid.

[8] The energy dissipation rate due to viscosity (per unit mass) is given by the work rate due to the viscosity term $\nu \triangle \boldsymbol{v}$ in the Navier-Stokes equations, which is estimated as $\boldsymbol{v} \cdot (\nu \Delta \boldsymbol{v}) \sim \nu v_l^2/l^2$.

4 Self-gravitating Fluids

4.1 Free fall

- In §3.1, we described the hydrostatic equation of a spherically symmetric self-gravitating object. Here, we will consider the gravitational contraction of spherically symmetric objects where the pressure is negligible, i.e., free fall contraction. Let us examine how a fluid particle located at the position r_0 at the initial time $t = 0$ falls toward the center. It is assumed that each part of the object is at rest initially. Fluid particles are accelerated by the self-gravity of the object. The gravitational field in a spherically symmetric object is given by Eq. (3.4) using $M(r)$ of Eq. (3.5). Since the surrounding fluid also falls with the fluid particle without passing, the mass $M(r)$ inside this particle is constant even at time $t > 0$ and equal to $M(r_0)$. Therefore, the Lagrangian equation of motion of this fluid particle is given by

$$\frac{d^2 r}{dt^2} = -\frac{GM(r_0)}{r^2} \tag{4.1}$$

- Energy integral. Multiplying both sides of this equation by the velocity dr/dt and integrating over time, we obtain

$$\frac{dr}{dt} = -\sqrt{2GM(r_0)\left(\frac{1}{r} - \frac{1}{r_0}\right)}, \tag{4.2}$$

  where it is used that dr/dt is zero at the initial position $r = r_0$ and negative for $t > 0$ (or $r < r_0$).

- The differential equation (4.2) can be readily integrated. In fact, with the variable transformation of $r/r_0 = \cos^2\theta$, Eq. (4.2) becomes

$$2\cos^2\theta \, \frac{d\theta}{dt} = \sqrt{\frac{2GM(r_0)}{r_0^3}}. \tag{4.3}$$

  Noting that $\theta = 0$ at $t = 0$, we can integrate this equation as

$$\theta + \frac{1}{2}\sin 2\theta = \sqrt{\frac{2GM(r_0)}{r_0^3}}\, t. \tag{4.4}$$

  This equation gives r as a function of time t, with the parameter θ.

- Using the above solution, the time $t_{\rm fall}$ required for this fluid particle to fall to the center is given by.

$$t_{\rm fall} = \frac{\pi}{2}\sqrt{\frac{r_0^3}{2GM(r_0)}}. \tag{4.5}$$

The time interval t_{fall} is called the free fall time. The initial average density within r_0, $\bar{\rho}(r_0)$, is defined by

$$\bar{\rho}(r_0) = \frac{M(r_0)}{4\pi r_0^3/3}. \tag{4.6}$$

Using this, the free fall time t_{fall} is rewritten as

$$t_{\mathrm{fall}} = \sqrt{\frac{3\pi}{32G\bar{\rho}(r_0)}}. \tag{4.7}$$

4.2 Jeans instability

- Consider a gas at rest with uniform density and pressure. Assume that self-gravity does not work when the gas is uniform and isotropic[9]. That is, ρ_0, p_0, ϕ_0= constant, and $\boldsymbol{v}_0 = 0$.We examine this self-gravitational instability of a uniform gas using the linear stability analysis described in the previous chapter. This problem is called the **Jeans instability**.
- Write perturbations of each quantity as ρ_1, p_1, ϕ_1, $\boldsymbol{v}_1$, and assume them adiabatic. Since the unperturbed state is uniform, these perturbations have a coordinate and time dependence of $\exp\left[\, i(\boldsymbol{k}\cdot\boldsymbol{x} - \omega t)\,\right]$.
- The first-order perturbation equation for each equation is obtained as

$$\text{Equation of continuity} \qquad -i\omega\rho_1 + i\rho_0\,\boldsymbol{k}\cdot\boldsymbol{v}_1 = 0, \tag{4.8}$$

$$\text{Euler's equation} \qquad -i\omega\boldsymbol{v}_1 = -i\boldsymbol{k}\,(c_s^2\frac{\rho_1}{\rho_0} + \phi_1), \tag{4.9}$$

$$\text{Poisson's equation} \qquad -k^2\phi_1 = 4\pi G\rho_1. \tag{4.10}$$

Eliminating $\boldsymbol{v}_1$, ϕ_1 from these equations, we obtain the dispersion relation as

$$\omega^2 = c_s^2\,k^2 - 4\pi G\rho_0. \tag{4.11}$$

Therefore, we find that the perturbations with wavenumbers satisfying

$$k \;<\; k_J \;\equiv\; \frac{\sqrt{4\pi G\rho_0}}{c_s} \tag{4.12}$$

have a negative ω^2 and grow exponentially, and as a result, the self-gravitational contraction proceeds.

[9]This assumption of gravitational equilibrium in the unperturbed state is not correct. That is, the gravitational equilibrium would not be reached without the pressure gradient and other effects that balance with gravity. This flaw is referred to as "the Jeans swindle." Nevertheless, the results of the simple Jeans instability are useful for understanding self-gravitational instabilities in real systems that are in equilibrium with other effects.

- Properties of a gaseous cloud collapsing due to Jeans instability

  - Jeans length
$$\lambda_J \equiv \frac{2\pi}{k_J} = \sqrt{\frac{\pi c_s^2}{G\rho_0}}. \tag{4.13}$$

  - Collapsing time $\simeq 1/(k_J c_s) = 1/\sqrt{4\pi G\rho_0}$ ($\sim$ free fall time.)

  - Jeans mass
$$M_J \simeq \frac{4\pi}{3}\rho_0 \left(\frac{\lambda_J}{2}\right)^3 \quad \propto \rho_0^{-1/2}. \tag{4.14}$$

Problem 19. Derive Eqs. (4.8)-(4.11).

Problem 20. Find the Jeans length [pc], the Jeans mass [$M_\odot$], and the collapsing time [yr] for a molecular cloud with temperature of 10K and density of $n_{H_2} = 50$ [cm^{-3}].

4.3 Virial theorem

- Let us derive a relation called the virial theorem that holds for spherically symmetric self-gravitating objects in hydrostatic equilibrium such as stars. As mentioned in §3.1, to determine the hydrostatic structure of a self-gravitating object, in addition to the hydrostatic equation, we need the energy equation that determines the temperature distribution. On the other hand, the virial theorem holds universally regardless of the mode of energy transport.

- To derive the virial theorem, we start with the hydrostatic equation (3.6) for a spherically symmetric self-gravitating object. Multiplying both sides of this equation by r and integrating over the entire volume of the object, we obtain
$$\int_0^R \frac{dp}{dr}\, r\, 4\pi r^2 dr = -\int_0^R \frac{GM(r)\rho}{r^2}\, r\, 4\pi r^2 dr, \tag{4.15}$$
where R is the radius of the object's surface where $\rho = p = 0$. With the integration by parts, the left-hand side of this equation becomes
$$\int_0^R \frac{dp}{dr}\, 4\pi r^3 dr = \left[p\ 4\pi r^3\right]_0^R - 3\int_0^R p\ 4\pi r^2 dr = -3\int_0^R (\gamma-1)\,\rho e\, 4\pi r^2 dr = -3\,(\gamma-1)\,U. \tag{4.16}$$
and can be expressed by the total internal energy U. In the second and third equalities in the above equation, we assume the ideal gas with a constant specific heat. On the other hand, the right-hand side of Eq. (4.15) is transformed as
$$-\int_0^R \frac{GM(r)\rho}{r}\, 4\pi r^2 dr = -\int_0^{M(R)} \frac{GM(r)}{r}\, dM(r) \tag{4.17}$$

and equal to the total gravitational energy of the object. Then, we obtain the **virial theorem**[10].

$$3\,(\gamma-1)\,U+W=0. \tag{4.18}$$

Problem 21. Generally, the total gravitational energy of an object is given by

$$W\equiv-\frac{1}{2}\int_V\int_V\frac{G\rho(\boldsymbol{x})\rho(\boldsymbol{x}')}{|\boldsymbol{x}-\boldsymbol{x}'|}\,d^3x\,d^3x'=\frac{1}{2}\int_V\rho(\boldsymbol{x})\phi(\boldsymbol{x})\,d^3x. \tag{4.19}$$

For a spherically symmetric object of radius R, show that this definition is equal to the right-hand side of Eq. (4.17). (Hint) Integrate it by parts and use $d\phi/dr = GM(r)/r^2$.

- Using the virial theorem, the total energy U + W of a hydrostatic object is rewritten as

$$U+W=(4-3\gamma)U. \tag{4.20}$$

  Since U is positive, if $\gamma > 4/3$, the total energy is negative, and the object is bounded by the self-gravity. Conversely, if $\gamma < 4/3$, the object is unbounded.

- Suppose an object bound by the self-gravity with $\gamma > 4/3$ emits radiation from its surface. Then its total energy decreases, and the absolute value of the total gravitational energy $|W|$, the total internal energy U, and the average temperature of the star increase. In other words, self-gravitating objects have the peculiar property that their average temperature increases when they release energy. Therefore, we can say that **self-gravitating objects have a negative heat capacity**.

4.4 Hydrostatic structures of polytropic gas spheres

- We solve the hydrostatic equation for spherically symmetric gaseous objects under simple assumptions. Differentiating the hydrostatic equation (3.6) multiplied by r^2/ρ, we have

$$\frac{d}{dr}\left(r^2\frac{1}{\rho}\frac{dp}{dr}\right)=-4\pi G\rho\,r^2. \tag{4.21}$$

  Although we usually need the energy equation governing the temperature distribution $T(r)$ to solve Eq. (4.21), we assume the polytropic relation

$$p=K\rho^{\Gamma},\qquad (K\text{ and }\Gamma\text{ are constant}) \tag{4.22}$$

  for simplicity. Note the exponent Γ is generally different from γ. The polytropic index $n = 1/(\Gamma-1)$ is often used instead of Γ. The hydrostatic sphere that satisfies

[10]Since the static object is considered here, we used Euler's equation, setting $\partial\boldsymbol{v}/\partial t+\boldsymbol{v}\cdot\nabla\boldsymbol{v}=0$. The virial theorem including these terms gives the more general relation, which is also valid for the object with internal motion.

this relation is called a polytropic gas sphere. For an ideal gas, the polytropic relation gives $T(r) \propto \rho(r)^{1/n}$, and the larger n is, the closer to isothermal.

Problem 22. A real hydrostatic gaseous object is stable (or marginally stable) against convection and satisfies $ds/dr \geq 0$. From this, show that $\Gamma \leq \gamma$ for gas spheres.

- **Lane-Embden equation**: Density and pressure are expressed with the dimensionless parameter θ as

$$\rho = \rho_c \theta^n, \qquad p = p_c \theta^{n+1}. \tag{4.23}$$

These expressions satisfy the polytoropic relation with $\Gamma = 1 + 1/n$. Substituting them into Eq. (4.21) yields the differential equation for θ

$$\left[\frac{(n+1)p_c}{4\pi G \rho_c^2}\right] \frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{d\theta}{dr}\right) = -\theta^n. \tag{4.24}$$

The inside of [ ] in the above equation has a dimension of length squared. Using the length a defined by

$$a = \left[\frac{(n+1)p_c}{4\pi G \rho_c^2}\right]^{1/2}, \tag{4.25}$$

we introduce the dimensionless radial coordinate ξ normalized as

$$\xi = r/a. \tag{4.26}$$

Equation (4.24) is expressed with ξ instead of r as

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2 \frac{d\theta}{d\xi}\right) = -\theta^n. \tag{4.27}$$

This is called the **Lane-Embden equation** and determines the hydrostatic structure of polytropic gas spheres.

- The variable θ satisfies the boundary conditions at $\xi = 0$

$$\theta = 1, \qquad \frac{d\theta}{d\xi} = 0 \qquad (\xi = 0). \tag{4.28}$$

The former is clear from Eq. (4.23). The latter is because the pressure gradient at the center vanishes from Eq. (4.21). The solution to Eq. (4.27) that also satisfies these boundary conditions is called the **Lane-Embden function** and denoted by θ_n. Figure 11 shows the Lane-Embden function for some polytropic indices n. The Lane-Embden function θ_n decreases monotonically from the center and vanishes at the surface of the object. The radial coordinate of the surface is denoted by ξ_1, and the radius of the object R is given by $\xi_1 a$. The dimensionless radius ξ_1 increases with n and is divergent at $n = 5$. Table 1 lists the constants for the Lane-Emden function, including ξ_1.

- The total mass of the object $M(R)$ is given by

$$\frac{M(R)}{4\pi a^3 \rho_c} = \int_0^{\xi_1} \theta^n \xi^2 d\xi = -\int_0^{\xi_1} \frac{d}{d\xi}\left(\xi^2 \frac{d\theta}{d\xi}\right) d\xi = -\left(\xi^2 \frac{d\theta}{d\xi}\right)_{\xi=\xi_1}. \tag{4.29}$$

In the second equality, the Lane-Embden equation is used. The average density $\bar{\rho}$ is given by

$$\frac{\bar{\rho}}{\rho_c} = \frac{M(R)}{4\pi R^3 \rho_c/3} = -\frac{3}{\xi_1}\left(\frac{d\theta}{d\xi}\right)_{\xi=\xi_1}. \tag{4.30}$$

Problem 23. For $n = 0$, 1, and 5, show that the Lane-Embden functions are given by

$$\theta_0 = 1 - \frac{1}{6}\xi^2, \qquad \theta_1 = \frac{\sin\xi}{\xi}, \qquad \theta_5 = \left(1 + \frac{\xi^2}{3}\right)^{-1/2}. \tag{4.31}$$

Problem 24. Show that the total gravitational energy of a polytropic gas sphere with $n < 5$ is given by

$$W = \frac{1}{2}\int \rho\phi\, dV = -\frac{3}{5-n}\frac{GM^2}{R} \qquad (n < 5\text{ の場合}). \tag{4.32}$$

To do so, firstly derive the relation

$$(n+1)\frac{p}{\rho} + \phi = -\frac{GM}{R} \quad \text{(constant)}. \tag{4.33}$$

Using this equation and the virial theorem $W = -3\int_0^R pdV$, derive Eq. (4.32). Furthermore, if p_c and ρ_c are finite, show that the total mass of a polytropic gas sphere with $n = 5$ is also finite, and derive the expression of W

$$W = -\frac{\sqrt{3}\pi}{32}\frac{GM^2}{a} \qquad (\text{for } n = 5). \tag{4.34}$$

If necessary, use the integral formula $\int_0^\infty \frac{x^2 dx}{(1+x^2)^3} = \pi/16$. Comparing Eq. (4.32) with (4.34), we find that ξ_1 is approximately given by $\frac{32\sqrt{3}}{\pi(5-n)} \fallingdotseq \frac{17.6}{5-n}$ when n is sufficiently close to 5.

Problem 25. For $n = 3.4, 4.9, 4.99$, solve numerically the Lane-Emden equation and find the values of ξ_1 and $-\xi_1^2(d\theta/d\xi)_{\xi=\xi_1}$ to the 6th digit accurately. Also, briefly describe the program you used.

Problem 26. The Lane-Emden function with $n = 3$ is used for a simple estimate of the stellar structure. Using the solar mass (1.99×10^{30}kg), the solar radius (6.96×10^5km), and θ_3, estimate the density and pressure at the center of the sun, and find the temperature at the center, using the average molecular weight of 0.61. (In the standard solar model of Bahcall et al. (1995), $\rho_c = 160\text{g/cm}^3$, $p_c = 2.4 \times 10^{16}$Pa, $T_c = 1.6 \times 10^7$K.) Also, show that the radiation pressure $p_{rad} = \frac{4\sigma}{3c}T^4$ is negligible inside the sun. ($\sigma = 5.7 \times 10^{-8}\text{W/m}^2/\text{K}^4$, $c = 3.0 \times 10^8$m/s.)

Problem 27. The maximum mass of a white dwarf, "Chandrasekhar limit". After the end of hydrogen burning or helium burning, low- or intermediate-mass stars complete their fusion and evolve into **white dwarfs**. The interior of a white dwarf is extremely dense (about 10^7 g/cm^3), and the structure is supported by the degeneracy pressure of electrons generated by such high density. The degeneracy pressure of the completely degenerate electrons is approximately given by the power of the density, as in the polytropic relation. The polytropic exponent n is 1.5 at a relatively low density and gradually increases with the density, reaching 3 in the high-density limit. In this high-density limit, the coefficient K of the polytropic relation is given by[11]

$$K = \frac{1}{8}\left(\frac{3}{\pi}\right)^{1/3} \frac{hc}{(m_u \mu_e)^{4/3}}, \tag{4.35}$$

where h is the Planck constant, c is the speed of light, m_u is the atomic mass unit, and μ_e is the number of nucleons per electron.

- For polytropes where the coefficient K is independent of the central density ρ_c, express the total mass M as a function of ρ_c and K, and show that the total mass does not depend on the central density when $n = 3$.

- As the mass of the white dwarf increases, the central density and central pressure increase to support the structure. Using the above result, show that the maximum mass that can be supported by the electron degeneracy pressure is given by

$$M_{\max} = \sqrt{\frac{3}{2}}\, \frac{2.0182}{4\pi\,(m_u \mu_e)^2} \left(\frac{hc}{G}\right)^{3/2}. \tag{4.36}$$

  Also, find the value of $M_{\max}$ for $\mu_e = 2.0$, and explain why $\mu_e = 2$.

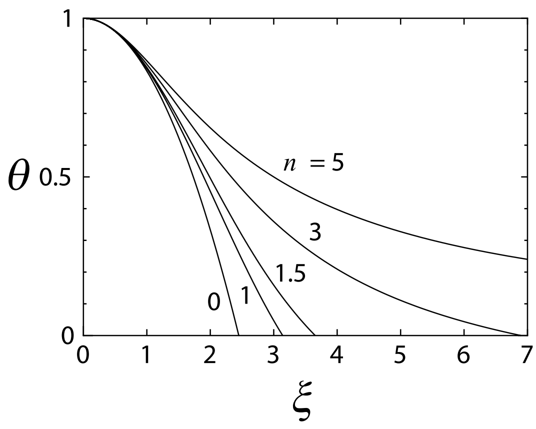


Figure 11: Lane-Emden functions.

Table I: Constants for the Lane-Emden function

n	ξ_1	$-\xi_1^2(d\theta/d\xi)_{\xi=\xi_1}$	$\rho_c/\bar{\rho}$
0	$\sqrt{6}$ = 2.45	$2\sqrt{6}$ = 4.90	1
1	π = 3.14	π = 3.14	$\pi^2/3$ = 3.29
1.5	3.6538	2.7141	5.9907
3	6.8968	2.0182	54.182
5	∞	$\sqrt{3}$ = 1.73	∞

[11] The Fermi momentum p_F of electrons with the number density n_e is given by $p_F \sim h n_e^{1/3}$. In the high-density limit (i.e., in the ultrarelativistic limit), the Fermi energy is $\epsilon_F = c p_F$, and the electron degeneracy pressure is obtained by differentiating ϵ_F by the volume per electron, $1/n_e$. Therefore, we obtain the pressure as $p \sim hc n_e^{4/3}$, and using $n_e = \rho/(m_u \mu_e)$ the coefficient K is estimated to be $\sim hc/(m_u \mu_e)^{4/3}$.

This upper limit on the mass of a white dwarf is called the **Chandrasekhar limit**. A white dwarf whose mass exceeds the Chandrasekhar limit due to gas accretion can no longer maintain its hydrostatic structure and shrinks rapidly, followed by a Type Ia supernova explosion.

(ASIDE) Hydrostatic structure of neutron stars

Neutron stars are the densest objects except for black holes and are fomed by supernovae. The central density of a neutron star with a solar mass is about $10^{15}\mathrm{g/cm^3}$ (about the same as the internal density of nuclei), and its radius is about 10 km. Because such dense neutron stars have extremely strong self-gravity, it is necessary to use general relativity rather than Newtonian mechanics to investigate their hydrostatic structure. The hydrostatic equation for a spherically symmetric object derived from the general theory of relativity is called the **TOV equation** (**Tolman-Oppenheimer-Volkoff equation**) and is given by[12]

$$\frac{dp}{dr} = -\frac{G\left(M(r) + 4\pi r^3 p/c^2\right)\left(\rho + p/c^2\right)}{r\left(r - 2GM(r)/c^2\right)}. \tag{4.37}$$

Using the TOV equation with the equation of state $p = p(\rho)$, we can derive the upper mass limit of neutron stars. Since the relativistic effects enhance gravity, the upper mass limit of neutron stars obtained from the TOV equation is much smaller than the limit of several solar masses obtained from Eq. (4.36) based on Newtonian mechanics.

[12]See §8.5 for the derivation. We can see the effects of the general relativity by comparing Eq. (3.6) with the Newtonian hydrostatic equation (3.6). All the relativistic effects on the right-hand side in Eq. (3.6) work to enhance gravity.

5 Spherically Symmetric Flow

5.1 Stellar wind

Stellar wind or solar wind is the supersonic outflow of gas from the surface of a star. The stellar wind is an example of spherically symmetric compressible steady flow, and the transition to supersonic velocity described in §2.6 is applicable.

(a) Hydrostatic model of the stellar atmosphere

- Before considering the flow of the stellar wind, we examine the hydrostatic structure of a stellar atmosphere is described. Assuming a corona, we consider an isothermal atmosphere. It is convenient to use the isothermal sound velocity $c_s^2 = k_B T/m$ in an isothermal atmosphere. The solar corona has a high temperature of over 1 million K and a sound velocity of over 100 km/sec.

- Substituting $\rho = p/c_s^2$ into the hydrostatic equation and integrating it, we have

$$c_s^2 \ln\left(\frac{p}{p_0}\right) = GM_{\rm star}\left(\frac{1}{r} - \frac{1}{R}\right). \tag{5.1}$$

  where $c_s = (k_B T/m)^{1/2}$ is the isothermal sound velocity, p_0 is the pressure at the stellar surface, and R is the stellar radius. This equation gives the pressure at infinity, $p(\infty)$, as

$$\frac{p(\infty)}{p_0} = \exp\left(-\frac{GM_{\rm star}}{c_s^2 R}\right). \tag{5.2}$$

- Using Eq. (5.2) and the temperature of the solar corona, we find that $p(\infty)$ is several orders of magnitude lower than p_0. However, the pressure in the interstellar medium near the Sun is still many orders of magnitude lower than this estimate of $p(\infty)$. Therefore, the solution of a hydrostatic stellar atmosphere is unrealistic. Instead, there is an outflow from the upper atmosphere to the outside, i.e., a stellar wind occurs.

(b) Parker's stellar wind solution

- Let us consider the steady stellar wind. Although a steady compressible flow is described by Bernoulli's equation, we start with Euler's equations, as in §2.6. Assume a spherically symmetric steady flow where $\boldsymbol{v} = (v(r), 0, 0)$ and v is the r component of the velocity. The r component of Euler's equation including gravity is given by

$$v\frac{dv}{dr} = -\frac{c_s^2}{\rho}\frac{d\rho}{dr} - \frac{GM_{\rm star}}{r^2}. \tag{5.3}$$

  On the other hand, the equation of continuity shows that the mass flux $S\rho v$ of a spherically symmetric steady flow is constant. Since $S = 4\pi r^2$ in this case, we have

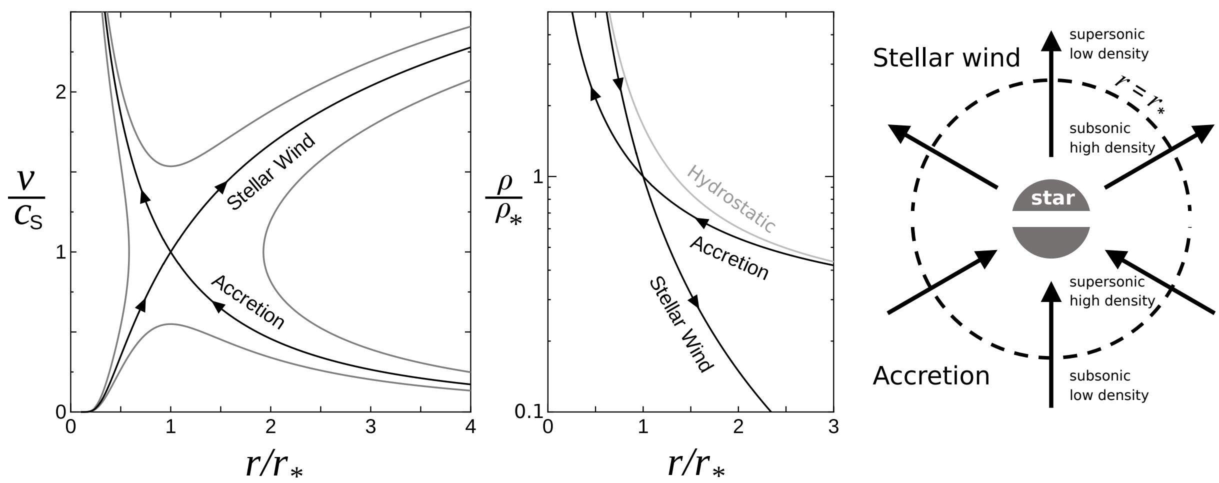


Figure 12: The left and middle panels display the velocity and density of Parker's isothermal solution, respectively. The right shows schematics of the stellar wind (top) and the accretion flow (bottom).

$$4\pi r^2 \rho v = \dot{M}_{\rm out} \quad \text{(const.)}, \tag{5.4}$$

where $\dot{M}_{\rm out}$ is the mass loss rate due to the outflow. Differentiating this by r yields

$$\frac{1}{\rho}\frac{d\rho}{dr} = -\frac{1}{v}\frac{dv}{dr} - \frac{2}{r}. \tag{5.5}$$

- Substituting Eq. (5.5) into (5.3), we obtain the stellar wind equation as

$$\left(\frac{v^2}{c_s^2} - 1\right)\frac{1}{v}\frac{dv}{dr} = \frac{2}{r} - \frac{GM_{\rm star}}{c_s^2\, r^2}. \tag{5.6}$$

This equation is similar to the equation (2.58) for the flow through a de Laval nozzle. The right-hand side is negative at small radius r and positive at large r (as is dS/dx of a de Laval nozzle). The critical radius r_* at which the right-hand side vanishes is given by

$$r_* = \frac{GM_{\rm star}}{2c_s^2}. \tag{5.7}$$

Therefore, the outflow velocity of the stellar wind is subsonic inside r_* and supersonic outside r_*. In the case of an isothermal flow, Eq. (5.6) can be easily integrated as

$$\frac{1}{2}\frac{v^2}{c_s^2} - \ln\left(\frac{v}{c_s}\right) = 2\ln\left(\frac{r}{r_*}\right) + 2\frac{r_*}{r} - \frac{3}{2}, \tag{5.8}$$

where the constant of integration, $-3/2$, is chosen so that $v = c_s$ at r_*. This solution can also be obtained from Bernoulli's equation. This steady solution of the isothermal stellar wind is called **Parker's solution**. Figure 12 shows Parker's solution.

- Assuming that the temperature in the solar corona is constant at 1.5 million K, the critical radius r_* of the solar wind is estimated from Eq. (5.7) to be about five times the solar radius. Also, from Eq. (5.6), in the isothermal case, the velocity gradient at the critical radius is obtained as $dv/dr = \pm c_s/r_*$. The sign on the right-hand side is positive for stellar winds.

- **Acceleration mechanism of stellar wind**
  - We first check the physical meanings of each term in Eq. (5.6) to understand the acceleration mechanism. Equation (5.6) originates from Euler's equation. The first term in parentheses on the left side is originally $(\boldsymbol{v} \cdot \mathbf{grad})\boldsymbol{v}$, which is the acceleration term in a steady flow. The second term on the left side and the first on the right side come from the pressure gradient term. The second term on the right side is the gravity term.
  - Outside the critical radius r_*, we see that the first terms on each side are larger than the others and balance each other. Therefore, from their meanings, we find that the acceleration is caused by the pressure gradient.
  - Inside the critical radius r_*, the second terms on each side balance. The two balancing terms are the pressure gradient and the gravity. That is, the gas is close to the hydrostatic equilibrium. Therefore, the acceleration here is not determined by the pressure gradient[13]. To keep the mass flux (5.4) constant, the velocity should increase as the density decreases rapidly in the downstream direction. The constant mass flux is driven by the pressure gradient near and outside the critical radius.

Problem 28. When the polytropic relation holds outside the critical radius, show that the velocity of the stellar wind sufficiently far away from r_* (i.e., the terminal velocity) is given by

$$v(\infty) = \sqrt{\frac{5-3\Gamma}{\Gamma-1}}\, c_{s,*} = \sqrt{\frac{5-3\Gamma}{\Gamma-1}\frac{GM_{\rm star}}{2r_*}}. \tag{5.9}$$

5.2 Accretion

- Another example of a spherically symmetric steady flow is a flow of interstellar gas towards a gravitational source such as a star. (see the lower right panel of Figure 12). The gravitational source is at rest in the surrounding gas. This flow is called the accretion. Although the radial velocity of the accretion is negative, Eqs. (5.3)-(5.6) apply similarly to stellar winds. Therefore, the accretion flow also has the critical radius r_* of Eq. (5.7), where $|v| = c_s$. Since the outside is upstream in the accretion flow, the flow is subsonic outside the critical radius and supersonic inside it.

[13]The acceleration inside r_* is directly caused by the pressure gradient slightly exceeding gravity. Nevertheless, this small imbalance between the pressure gradient and gravity is determined so that the mass flux remains constant and is considered as the secondary factor.

- The quantities at r_* are estimated using Bernoulli's equation (5.10). In this section, we assume that the polytropic relation holds in the accretion flow. Then Bernoulli's equation is written as[14]

$$\frac{1}{2}v^2 + \frac{c_s^2}{\Gamma - 1} - \frac{GM_{\rm star}}{r} = \frac{c_{s,0}^2}{\Gamma - 1}, \tag{5.10}$$

where we used $c_s^2 = \Gamma p/\rho$, and $c_{s,0}$ is the sound velocity far enough away from the gravitational source object. Writing the left-hand side of this equation with quantities at r_*, we have

$$c_{s,*} = \sqrt{\frac{2}{5 - 3\Gamma}}\, c_{s,0}. \tag{5.11}$$

Note that even monoatomic gas satisfies $\Gamma < 5/3$ and has finite $c_{s,*}$ when the radiative cooling of the gas is effective. The critical radius r_* and the density there ρ_* are given by

$$r_* = \frac{5 - 3\Gamma}{2}\frac{GM_{\rm star}}{2c_{s,0}^2}, \qquad \frac{\rho_*}{\rho_0} = \left(\frac{c_{s,*}}{c_{s,0}}\right)^{\frac{2}{\Gamma-1}} = \left(\frac{2}{5 - 3\Gamma}\right)^{\frac{1}{\Gamma-1}}. \tag{5.12}$$

where ρ_0 is the gas density far away.

- The mass accretion rate $\dot{M}_{\rm in}$ onto the object is defined by

$$\dot{M}_{\rm in} = -4\pi r^2 \rho v, \tag{5.13}$$

and independent of r. Expressing this with quantities at r_* and using the above equation, we obtain the mass accretion rate as

$$\dot{M}_{\rm in} = 4\pi r_*^2 \rho_* c_{s,*} = 4\pi \left(\frac{2}{5 - 3\Gamma}\right)^{\frac{5-3\Gamma}{2(\Gamma-1)}} \left(\frac{GM_{\rm star}}{2c_{s,0}^2}\right)^2 \rho_0\, c_{s,0} \tag{5.14}$$

The above solution and the mass accretion rate are called the Bondi solution and the Bondi accretion rate, respectively. Using the Bondi radius r_B defined by $2GM_{\rm star}/c_{s,0}^2$, the Bondi accretion rate is often simply estimated as $4\pi r_B^2 \rho_0\, c_{s,0}$, neglecting its Γ dependence.

Problem 29. Explain the acceleration mechanism of the accretion flow inside and outside r_* based on Eq. (5.6) as in the case of the stellar wind[15].

[14] In Chapter 1, we derived Bernoulli's equation for the adiabatic flow. Equation (5.10) is valid even when the polytropic relation holds instead of the adiabatic condition. Noting that $c_s^2 = \Gamma p/\rho$ and differentiating Eq. (5.10) by r, we obtain Euler's equation (5.3).

[15] The supersonic jet formed in the de Laval nozzle can be explained in the same way as these flows.

5.3 Blast waves

(a) Model of a blast wave

We examine a blast wave generated by a point source explosion and its propagation in the uniform gaseous medium. The blast wave is the flow of the gaseous medium blown out by the explosion. Due to the following assumptions, the gas flow is determined only by the explosion energy E and the density of the surrounding medium ρ_0. The flow is spherically symmetric for the point source.

- The point source explosion is assumed. The mass, momentum, and initial volume of the gas released by the explosion are all negligibly small. The explosion duration is sufficiently short.
- The initial pressure, sound velocity, and internal energy of the surrounding medium can be ignored due to the strong explosion.
- Cooling due to radiation and gravity can be ignored.

An analytical solution has been obtained for the ideal blast wave, known as **Sedov's solution**[16].

(b) Dimensional analysis of blast waves

By dimensional analysis, we can derive the following scaling laws for Sedov's solution without solving the hydrodynamical equations.

- In this problem, there exists only a dimensionless quantity $\dfrac{r^5 \rho_0}{t^2 E}$. Using it, each dimensional quantity is uniquely estimated.
- **Time evolution of a blast wave**
  - Shock front radius of a blast wave $\sim \left(\dfrac{E}{\rho_0}\right)^{1/5} t^{2/5}$
  - Propagation speed, flow velocity $\sim \left(\dfrac{E}{\rho_0}\right)^{1/5} t^{-3/5}$
  - Density $\sim \rho_0$, Pressure $\sim \rho_0 \left(\dfrac{E}{\rho_0}\right)^{2/5} t^{-6/5}$ $(\gg p_0 = \rho_0 c_{s,0}^2/\gamma)$.
- **Structure of a blast wave**
  - The front of a blast wave is a spherical shock wave with a Mach number $\gg 1$. Outside the shock wave, the gaseous medium is still at rest and uniform.

[16] This analytical solution was published by Sedov in 1946. The numerical solution was reported earlier by Taylor.

– Inside the shock wave, the spatial distributions of the velocity, density, and pressure depend only on a dimensionless coordinate given by

$$\xi = \left(\frac{\rho_0}{E\,t^2}\right)^{1/5} r. \tag{5.15}$$

Therefore, the spatial distributions are similar at all times. Such a solution is called a **similarity solution**[17].

– The position of the shock wave is written with the dimensionless coordinate as $\xi = \xi_s$ (constant) and with the dimensionless coordinate and expressed with the r-coordinate as

$$r = r_s(t) \equiv \xi_s \left(\frac{E\,t^2}{\rho_0}\right)^{1/5}. \tag{5.16}$$

Propagation speed of the shock wave is

$$v_s = \frac{d}{dt} r_s(t) = \frac{2r_s(t)}{5t} = \frac{2\xi_s}{5}\left(\frac{E}{\rho_0\,t^3}\right)^{1/5}. \tag{5.17}$$

The constant ξ_s is found by solving the hydrodynamic equations.

- **Applicable range of Sedov's solution.**

– The ignorance of the mass M of the gas released from the explosion requires the condition[18]

$$M \ll \frac{4\pi}{3} r_s(t)^3 \rho_0. \tag{5.18}$$

– To ignore the pressure of the surrounding medium, the condition

$$p_0 \ll \rho_0 \left(\frac{E}{\rho_0\,t^3}\right)^{2/5}, \qquad \text{or} \qquad c_{s,0} \ll \left(\frac{E}{\rho_0\,t^3}\right)^{1/5}. \tag{5.19}$$

is required.

These conditions give the lower and upper limits for the time t and r_s that bound the applicable range of Sedov's solution.

Problem 30. A supernova explodes, ejecting 10 solar masses of material with an energy of 10^{44}J, forming a blast wave. The surrounding interstellar gas is an atomic hydrogen gas with a number density of 1cm^{-3} and a temperature of 100K. In this case, estimate both lower and upper limits in radius r_s [pc] and time t [years] for the applicable range of Sedov's solution using Eqs. (5.18) and (5.19).

[17] The rarefaction wave in §2.2 is also a similarity solution. Its dimensionless coordinate is $x/(c_{s,0}t)$.

[18] The opposite limit of Eq. (5.18) corresponds to the initial free expansion phase, where the mass of the surrounding medium is negligible. At this phase, the released gas expands freely at a constant velocity.

(c) Derivation of Sedov's solution

- Dimensionless variables in the similarity solution $v'(\xi)$, $\rho'(\xi)$, $p'(\xi)$.

  The relations between the variables v, ρ, p, and the dimensionless variables are given by

$$v(r,t) = \frac{2r}{5t}\, v'(\xi), \qquad \rho(r,t) = \rho_0\, \rho'(\xi), \qquad p(r,t) = \rho_0 \left(\frac{2r}{5t}\right)^2 p'(\xi), \tag{5.20}$$

  respectively. The time derivative and the r derivative for the dimensionless variables are expressed by the ξ derivative as

$$\frac{\partial}{\partial t}\rho'(\xi) = \frac{\partial \xi}{\partial t}\frac{d\rho'}{d\xi} = -\frac{2\xi}{5t}\frac{d\rho'}{d\xi}, \qquad \frac{\partial}{\partial r}\rho'(\xi) = \frac{\partial \xi}{\partial r}\frac{d\rho'}{d\xi} = \frac{\xi}{r}\frac{d\rho'}{d\xi} \tag{5.21}$$

- In the spherically symmetric case, the equation of continuity and the adiabatic condition are written as

$$\frac{\partial \rho}{\partial t} + \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 \rho v) = 0, \qquad \left(\frac{\partial}{\partial t} + v\frac{\partial}{\partial r}\right)\frac{p}{\rho^\gamma} = 0, \tag{5.22}$$

  where we used $p/\rho^\gamma = Ae^{s/c_V}$ (Eq. [1.65]). Substituting Eq. (5.20) into these equations and using Eq. (5.21), we obtain ordinary differential equations

$$\frac{dv'}{d\ln\xi} + (v'-1)\frac{d\ln\rho'}{d\ln\xi} + 3v' = 0, \qquad (v'-1)\frac{d\ln(p'/\rho'^\gamma)}{d\ln\xi} + 2v' - 5 = 0. \tag{5.23}$$

  The sum of these two equations can easily be integrated, and we obtain the so-called **adiabatic integral** as[19].

$$\frac{d}{d\ln\xi}\ln\left[\,\xi^5(v'-1)\,p'\rho'^{1-\gamma}\,\right] = 0, \qquad \text{and} \qquad \xi^5(v'-1)\,p'\rho'^{1-\gamma} = \text{const.} \tag{5.24}$$

- **Energy integral**

  Using the equation of energy conservation instead of Euler's equation, we will obtain another integral.

  - The total energy within the radius r is given by

$$E(r) = \int_0^r \varepsilon\, 4\pi r^2 dr, \tag{5.25}$$

    where $\varepsilon \equiv \rho(\frac{1}{2}v^2 + e)$ is the energy per volume, and e is the internal energy per unit mass. Reminding the energy conservation and the adiabatic condition and neglecting the initial internal energy of the medium, we find that $E(r_s(t)) = E$ (the explosion energy).

[19] The constant of the second equation of (5.24) is determined by the boundary conditions at $r = r_s$.

– Due to the similarity of the blast wave, $E(r)$ has the following property. The radius, $r(\xi)$, for a given ξ increases with time as

$$r(\xi) = \xi \left(\frac{Et^2}{\rho_0} \right)^{1/5}. \tag{5.26}$$

The energy ratio $E(r(\xi))/E$ defined by the radius $r(\xi)$ constant for time due to the similarity. That is, for a given ξ

$$\frac{d}{dt} E(r(\xi)) = 0. \tag{5.27}$$

– Taking the time derivative of Eq. (5.25), we obtain

$$\frac{d}{dt} E(r(\xi)) = \int_0^{r(\xi)} \frac{\partial \varepsilon}{\partial t} 4\pi r^2 dr + \varepsilon \, 4\pi r(\xi)^2 \frac{dr(\xi)}{dt}. \tag{5.28}$$

Using the equation of energy conservation (1.37), the first term on the right-hand side of the above equation is rewritten as

$$-\int_0^{r(\xi)} \frac{1}{r^2} \frac{d}{dr} \left[r^2 \rho v \left(\frac{1}{2} v^2 + h \right) \right] 4\pi r^2 dr = -\rho v \left(\frac{1}{2} v^2 + h \right) 4\pi r(\xi)^2. \tag{5.29}$$

Furthermore, the second term is transformed by $dr(\xi)/dt = 2r(\xi)/(5t)$ and since the sum of the terms vanishes, we obtain

$$4\pi r^2 \rho v \left(\frac{1}{2} v^2 + h \right) = 4\pi r^2 \frac{2r}{5t} \rho \left(\frac{1}{2} v^2 + e \right). \tag{5.30}$$

This equation is obvious. The left-hand side of this equation represents the energy flowing out of the radius $r(\xi)$ per unit time (the energy flux at $r(\xi)$) and the right-hand side represents the energy of the fluid entering the interior of $r(\xi)$ per unit time due to the increase in $r(\xi)$. These energies should be equal due to the invariance of $E(r(\xi))$.

– Noting that $e = \frac{1}{\gamma - 1} p/\rho$, $h = \frac{\gamma}{\gamma - 1} p/\rho$ for ideal gas, Eq. (5.30) is rewritten as

$$\frac{p}{\rho} = -\frac{\gamma - 1}{2} v^2 \frac{v - \frac{2r}{5t}}{\gamma v - \frac{2r}{5t}} \quad \text{or} \quad \frac{p'}{\rho'} = -\frac{\gamma - 1}{2} v'^2 \frac{v' - 1}{\gamma v' - 1}. \tag{5.31}$$

- **Boundary conditions at the shock wave**: The quantities (v_1, ρ_1, p_1) behind the shock are obtained from the Rankine–Hugoniot jump conditions for a strong shock ($p_1/p_0 \gg 1$).

  – The density is obtained from Eq. (2.38) as

$$\rho_1 = \frac{\gamma + 1}{\gamma - 1} \rho_0. \tag{5.32}$$

- Denoting the velocities in the rest frame of the shock in front of and behind the shock by u_0, $u_1\,(<0)$, respectively, and noting that $v_s = |u_0|$, we have

$$v_1 = u_1 - u_0 = \left(1 - \frac{|u_1|}{|u_0|}\right) v_s = \left(1 - \frac{\rho_0}{\rho_1}\right) v_s = \frac{2}{\gamma+1} v_s. \tag{5.33}$$

- From Eq. (2.43) for a strong shock, we have

$$u_0^2 = \frac{\gamma+1}{2\gamma}\frac{p_1}{p_0}\, c_{s,0}^2 = \frac{\gamma+1}{2}\frac{p_1}{\rho_0}, \tag{5.34}$$

and the pressure behind the shock is given by

$$p_1 = \frac{2}{\gamma+1}\,\rho_0\, v_s^2. \tag{5.35}$$

From Eqs. (5.32)-(5.35), the dimensionless variables at $\xi = \xi_s$ become

$$v'(\xi_s) = \frac{2}{\gamma+1}, \qquad \rho'(\xi_s) = \frac{\gamma+1}{\gamma-1}, \qquad p'(\xi_s) = \frac{2}{\gamma+1}. \tag{5.36}$$

- Using the above integrals and the boundary conditions, we can obtain the exact solution. From the integrals of Eqs. (5.24) and (5.31), we have

$$\xi^5\, \frac{v'^2\,(v'-1)^2}{\gamma v' - 1}\, \rho'^{\,2-\gamma} = \text{const.} \tag{5.37}$$

With this, the first equation of (5.23) can be transformed into the first-order differential equation for v'

$$-\frac{1}{(3\gamma-1)v' - 5}\left(\gamma + 1 + \frac{\gamma-1}{\gamma v' - 1} - \frac{2}{v'}\right)\frac{dv'}{d\ln\xi} = 1. \tag{5.38}$$

Furthermore, by the partial fraction decomposition, we obtain

$$\left(\frac{a}{v' - 5/(3\gamma-1)} + \frac{b}{v' - 1/\gamma} - \frac{2}{5v'}\right)\frac{dv'}{d\ln\xi} = 1, \tag{5.39}$$

where the coefficients a and b are given by

$$a = -\frac{13\gamma^2 - 7\gamma + 12}{5(3\gamma-1)(2\gamma+1)}, \qquad b = \frac{\gamma-1}{2\gamma+1}, \tag{5.40}$$

respectively. Equation (5.39) can be easily integrated, finally yielding the solution for v' as

$$\left[\frac{5-(3\gamma-1)v'}{(7-\gamma)/(\gamma+1)}\right]^a \left[\frac{\gamma v' - 1}{(\gamma-1)/(\gamma+1)}\right]^b \left[\frac{\gamma+1}{2}v'\right]^{-2/5} = \frac{\xi}{\xi_s}, \tag{5.41}$$

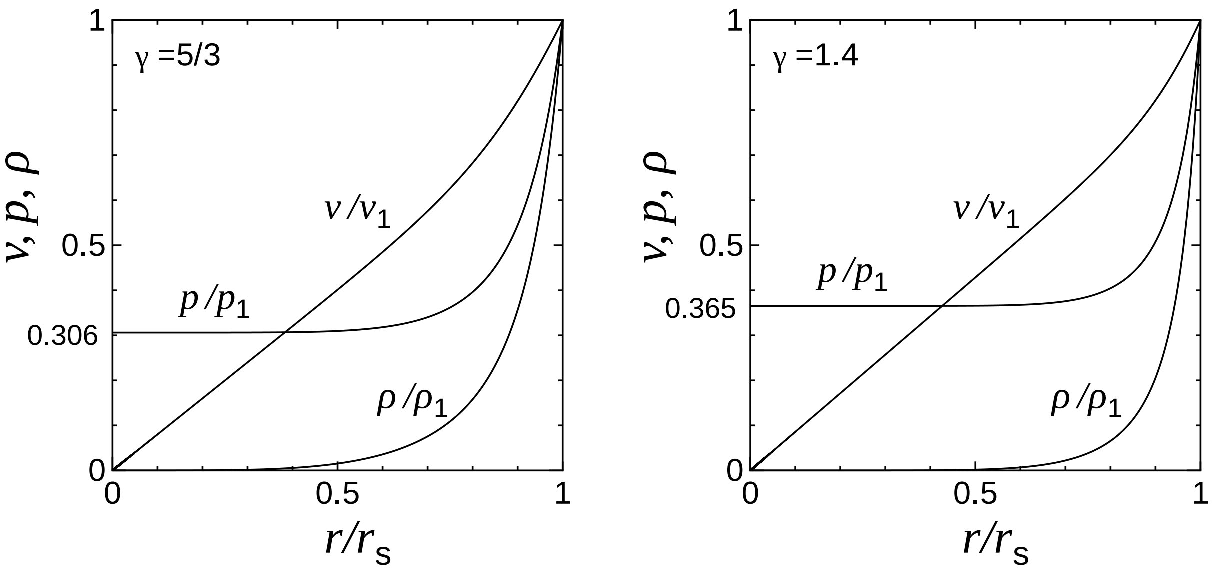


Figure 13: Sedov's solution for $\gamma = 5/3$ and 1.4. In these cases, the density ratios ρ_1/ρ_0 are 4 and 6, respectively, and the gas blown away by the blast wave is concentrated near the shock wave.

where the coefficients of each factor are determined such that the boundary condition $v' = 2/(\gamma+1)$ holds for $\xi = \xi_s$.

- Substituting Eq. (5.41) into (5.37) yields the density ρ' as the function of v'

$$\rho' = \frac{\gamma+1}{\gamma-1}\left[\frac{5-(3\gamma-1)v'}{(7-\gamma)/(\gamma+1)}\right]^{a'}\left[\frac{\gamma v'-1}{(\gamma-1)/(\gamma+1)}\right]^{b'}\left[\frac{1-v'}{(\gamma-1)/(\gamma+1)}\right]^{c'}, \quad (5.42)$$

where the coefficients are again determined by the boundary condition, and the indices a', b', c' are given by

$$a' = \frac{13\gamma^2 - 7\gamma + 12}{(3\gamma-1)(2\gamma+1)(2-\gamma)}, \qquad b' = \frac{3}{2\gamma+1}, \qquad c' = -\frac{2}{2-\gamma}. \quad (5.43)$$

Substituting this expression for ρ' into Eq. (5.31), we can also obtain the expression for p'. Figure 13 shows Sedov's solutions for $\gamma = 5/3$ and 1.4.

- The constant, ξ_s, is determined by the equation $E(r_s) = E$. Writing down Eq. (5.25) at $r = r_s$ with dimensionless variables and introducing a new variable $\chi = \xi/\xi_s$, we obtain

$$\xi_s^5 \frac{16\pi}{25}\int_0^1 \left(\frac{1}{2}\rho' v'^2 + \frac{p'}{\gamma-1}\right)\chi^4 d\chi = 1. \quad (5.44)$$

By numerically integrating this using the obtained exact solutions, we can obtain ξ_s. For example, $\xi_s = 1.1517$ and 1.0328 for $\gamma = 5/3$ and 1.4, respectively.

Problem 31. Sedov's solution near the center: Show that v' is close to $1/\gamma$ at $\xi \ll \xi_s$. Also, show that the velocity, density, pressure, and temperature each have the following r dependence near the center. (Use $\gamma < 7$.)

$$v \propto r, \qquad \rho \propto r^{3/(\gamma-1)}, \qquad p = \text{const.}, \qquad T \propto r^{-3/(\gamma-1)}. \tag{5.45}$$

Problem 32. Derive Eq. (5.44).

(ASIDE) Adiabatic integral

Let us see that the integral (5.24) comes from the entropy conservation. Similar to the cumulative energy distribution $E(r(\xi))$, we define the cumulative distribution of the entropy-related quantity p/ρ^γ ($\propto \mathrm{e}^{s/c_V}$) as[20]

$$Y(r(\xi)) = \rho_0^{\gamma-1} \int_{r(\xi)}^{r_s(t)} \rho \frac{p}{\rho^\gamma} 4\pi r^2 dr. \tag{5.46}$$

Since it has a dimension of energy, this integral is also independent of t due to the similarity as well as $E(r(\xi))$. Furthermore, Eq. (5.22) yields the equation equivalent to the enthalpy conservation

$$\frac{\partial}{\partial t}\left(\frac{p}{\rho^\gamma}\rho\right) + \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{p}{\rho^\gamma}\rho v\right) = 0. \tag{5.47}$$

Substituting this into the expression for the time derivative $dY(r(\xi))/d\ln t$, we obtain for a given ξ $(\leq \xi_s)$[21]

$$\left[4\pi r^2 \rho \left(\frac{2r}{5t} - v\right) \frac{p}{\rho^\gamma} t\right]_{r(\xi)}^{r_s(t)} = 0, \qquad \text{and} \qquad 4\pi r^2 \rho \left(\frac{2r}{5t} - v\right) \frac{p}{\rho^\gamma} t = \text{const.} \tag{5.48}$$

Writing down this integral with dimensionless variables yields Eq. (5.24)[22].

(d) Blast waves considering the cooling process

Although Sedov's solution neglects the cooling process, radiative cooling is effective in the later phase of a blast wave generated by a supernova explosion. Due to the cooling, Sedov's solution breaks down much earlier ($\sim 10^4$ years)[23] than the estimate in Problem 30. We describe the two subsequent phases of the blast wave when cooling is effective[24].

[20] Since the integrand diverges at the center, we take the integration range that does not include $r = 0$.

[21] From its derivation, the second equation of Eq. (5.48) indicates the independence of r. However, the left-hand side with the dimension of $E\rho_0^{1-\gamma}$ is also independent of t as shown by the dimension analysis.

[22] Using Eqs. (5.20) and (5.21), we can obtain Eq. (5.24) directly from the conservation equation (5.47) as in the derivation of Eq. (5.23).

[23] At this phase, the gas temperature is still several millions Kelvin and the gas is a fully ionized plasma. Such a hot fully ionized plasma cools down by line radiation of X rays, and the cooling rate is approximately given by $\Lambda = 1 \times 10^{-23}(n_\mathrm{H}/10^6\mathrm{m}^{-3})^2(T/10^6\mathrm{K})^{-0.7}$ [J/m^3/s] (see Fig. 1 of Gaetz & Salpeter 1983). The cooling time t_cool is estimated by the internal energy per unit mass of ionized gas, $3kT\rho/m_\mathrm{H}$, divided by the cooling rate Λ. Then, from the equation of $t = t_\mathrm{cool}$, the time at which the cooling becomes effective is obtained as $t = 3 \times 10^4 (E/10^{44}\mathrm{J})^{0.22}(n_\mathrm{H}/10^6\mathrm{m}^{-3})^{-0.55}$ years.

[24] The evolution of a supernova remnant is more complex due to non-uniform densities of the surrounding interstellar medium.

- **Pressure-driven snowplow phase**

  Radiative cooling is effective only in the high-density spherical shell near the shock wave. On the other hand, the inner low-density gas is not cooled and keeps high temperature and pressure. The cooled, dense spherical shell continues to expand, while being pushed by the high pressure inside. This is the **pressure-driven snowplow phase**.

  Let us estimate the expansion rate of the spherical shell at this phase with dimensional analysis. The inner gas expands adiabatically and satisfies $pV^\gamma =$ constant[25]. Since $V \propto r_s^3$, the inner pressure decreases as $p \propto r_s^{-3\gamma}$. Also, the inner pressure is approximately equal to that of the post-shock, $\rho_0 v_s^2$ (see Eq. [5.35]). Since $v_s \sim r_s/t$, we therefore find that the shell expands in the pressure-driven snowplow phase as

  $$r_s \propto t^{2/(3\gamma+2)}, \qquad v_s \propto t^{-3\gamma/(3\gamma+2)}, \qquad p \propto t^{-6\gamma/(3\gamma+2)}. \tag{5.49}$$

  Since $\gamma > 1$, the expansion in this phase is slower and the pressure decreases faster than those of Sedov's solution.

- **Momentum conserving snowplow phase**

  When the adiabatic expansion reduces the pressure sufficiently, the spherical shell is no longer pushed by the inside, and then the shell expands while conserving its momentum. This is the **momentum conserving snowplow phase**.

  The conserved momentum of a small part of the shell within a solid angle $\Delta\Omega$ is given by $\Delta M v_s$, where ΔM is the shell mass within $\Delta\Omega$ and increases as $\Delta M \propto r_s^3$. Thus, the expansion velocity v_s decreases as r_s^{-3}. Noting $v_s \sim r_s/t$ again, we find the shell expands in the momentum conserving snowplow phase as

  $$r_s \propto t^{1/4}, \qquad v_s \propto t^{-3/4}. \tag{5.50}$$

  Since $\gamma < 2$ usually, the expansion in this phase is slower than that in the pressure-driven snowplow phase. The expansion stops when the expansion velocity equals the sound velocity of the outer medium.

[25]Since the cooled shell is thin, the shock front radius r_s and the inner radius of the shell are almost equal.

6 Fundamentals of Magnetohydrodynamics

6.1 Basic equations of magnetohydrodynamics

(a) Electrically conducting fluids

- In space, we often deal with ionized gas (plasma). For example, the stellar interior and the HII region are highly ionized plasmas. The HI regions and molecular clouds with a low degree of ionization are called weakly ionized plasmas, Even the low degree of ionization cannot be ignored in many cases. Magnetohydrodynamics (MHD) is study describing the motion of plasmas. Since they have conductivity, the ionized fluids interact with the electromagnetic field. Thus, it is necessary to solve the hydrodynamics equations coupled with Maxwell's equations. Since we consider the macroscopic fluid, the mean free path of charged and neutral particles is assumed to be much smaller than the characteristic length of the fluid.

- Maxwell's equations (in SI units)

$$\nabla\cdot \boldsymbol{D} = \rho_e, \tag{6.1}$$

$$\nabla\cdot \boldsymbol{B} = 0, \tag{6.2}$$

$$\nabla\times \boldsymbol{E} + \frac{\partial \boldsymbol{B}}{\partial t} = 0 \qquad \text{(Faraday's law of induction)}, \tag{6.3}$$

$$\nabla\times \boldsymbol{H} - \frac{\partial \boldsymbol{D}}{\partial t} = \boldsymbol{j}_e \qquad \text{(the Ampère-Maxwell law)}. \tag{6.4}$$

  To convert the above equations to those in the often-used cgs-gauss units, we express $\boldsymbol{D}$ and $\boldsymbol{H}$ with $\boldsymbol{E}$ and $\boldsymbol{B}$, respectively, and perform the conversions, $\boldsymbol{B} \to \boldsymbol{B}/c$, $\varepsilon_0 \to 1/(4\pi)$, and $\mu_0 \to 4\pi/c^2$, This conversion formula from the SI to cgs-gauss is valid not only for Maxwell's equations but also for all equations in this section[26].

- Assumptions in MHD

  1. **Charge neutrality**: In magnetohydrodynamics, the charge density is negligible small.

  $$\rho_e \equiv \sum_i q_i n_i = 0, \qquad \text{c.f. charge current:} \quad \boldsymbol{j}_e \equiv \sum_i q_i n_i \boldsymbol{v_i} \neq 0. \tag{6.5}$$

     where n_i, q_i, and v_i are the number density, charge, and the mean velocities of the particle species i.

  2. **Non-relativistic case:** $v \ll c$

  3. **Ohm's law** (a steady current)

  $$\boldsymbol{j}_e = \sigma(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}), \tag{6.6}$$

[26] The vector potential $\boldsymbol{A}$, which is related with $\boldsymbol{B}$ as $\boldsymbol{B} = \nabla\times\boldsymbol{A}$ is converted as $\boldsymbol{A} \to \boldsymbol{A}/c$. The conversion from the SI to the cgs-gauss units is easy in this way, but the opposite conversion is not.

where σ is the conductivity. The Hall effect due to the magnetic effect is ignored here[27].

4. **Ampère's law**: In Eq. (6.4), the displacement current $\partial \boldsymbol{D}/\partial t$ ca be ignored, i.e.[28] [29],
$$\nabla \times \boldsymbol{H} = \boldsymbol{j}_e. \tag{6.7}$$

5. Other assumptions: The permittivity is set to be ε_0 and the magnetic permeability is μ_0 $(\varepsilon_0\mu_0 = 1/c^2)$. The electrical conductivity σ is constant in the fluid.

(b) Basic equations in MHD

- **Lorenz force on the electrically conducting fluid**

  The force exerted on a charged particle by the electromagnetic field is the Lorenz force,$q_i(\boldsymbol{E} + \boldsymbol{v}_i \times \boldsymbol{B})$. Therefore, the Lorenz force on the fluid per unit volume is given by
$$\sum_i n_i q_i(\boldsymbol{E} + \boldsymbol{v}_i \times \boldsymbol{B}) = \boldsymbol{j}_e \times \boldsymbol{B} = -\frac{1}{\mu_0}\boldsymbol{B} \times (\nabla \times \boldsymbol{B}), \tag{6.8}$$
  where the charge neutrality, the expression of the current density in Eq. (6.5), and Ampère's law (6.7) are used.

  This Lorenz force on the fluid is also expressed as
$$-\frac{1}{\mu_0}\boldsymbol{B} \times (\nabla \times \boldsymbol{B}) = -\nabla\left(\frac{|\boldsymbol{B}|^2}{2\mu_0}\right) + \frac{1}{\mu_0}(\boldsymbol{B}\cdot\nabla)\boldsymbol{B}, \tag{6.9}$$
$$\left[-\frac{1}{\mu_0}\boldsymbol{B} \times (\nabla \times \boldsymbol{B})\right]_i = \frac{\partial}{\partial x_j}\left(-\frac{B^2}{2\mu_0}\delta_{ij} + \frac{1}{\mu_0}B_iB_j\right). \tag{6.10}$$

  The first term on the right-hand side of Eq. (6.9)is the force due to the magnetic pressure $B^2/2\mu_0$ and the second term is the Maxwell stress tensor due to the magnetic field.

- **The equation of motion in MHD** is Euler's equation including the Lorenz force given by
$$\rho\left[\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{v}\right] = -\nabla p - \frac{1}{\mu_0}\boldsymbol{B} \times (\nabla \times \boldsymbol{B}). \tag{6.11}$$
  or

[27] The Hall effect can be neglected when $|\boldsymbol{v}_i - \boldsymbol{v}| \ll E/B$. The generalized Ohm's law is described in §6.4

[28] The condition for the displacement current to be negligible compared to $\boldsymbol{j}_e$ can be written as $T \gg \varepsilon_0/\sigma$ using Ohm's law, where T is the time in which the electromagnetic field changes according to Eq. (6.13).

[29] Although the charge current density $\boldsymbol{j}_e$ is determined by Ohm's law, Ampère's law is also satisfied at the same time. Therefore, these two expressions of $\boldsymbol{j}_e$ should be equivalent. The reason of the equivalence is explained as follows. If there is the difference $\Delta\boldsymbol{j}_e = \boldsymbol{j}_e - \nabla \times \boldsymbol{H}$, an additional component of the electric field oppoisite to $\Delta\boldsymbol{j}_e$ is generated by Eq. (6.4), and $\Delta\boldsymbol{j}_e$ always decreases due to Ohm's law. The decay time of $\Delta\boldsymbol{j}_e$ is estimated to be $\sim \varepsilon_0/\sigma$.

$$\rho\left[\frac{\partial \boldsymbol{v}}{\partial t}+(\boldsymbol{v}\cdot\nabla)\boldsymbol{v}\right]=-\nabla\left[p+\left(\frac{|\boldsymbol{B}|^2}{2\mu_0}\right)\right]+\frac{1}{\mu_0}(\boldsymbol{B}\cdot\nabla)\boldsymbol{B}. \tag{6.12}$$

- **Induction equation of the magnetic field**

  From Ampère's law (6.7) and Ohm's law (6.6), the electric field $\boldsymbol{E}$ is obtained as

$$\boldsymbol{E}=-\boldsymbol{v}\times\boldsymbol{B}+\frac{1}{\mu_0\sigma}\nabla\times\boldsymbol{B}. \tag{6.13}$$

  Taking the rotation of this equation and eliminating $\nabla\times\boldsymbol{E}$ from Eq. (6.3), we obtain the so-called induction equation as

$$\frac{\partial \boldsymbol{B}}{\partial t}=\nabla\times(\boldsymbol{v}\times\boldsymbol{B})+\frac{1}{\mu_0\sigma}\triangle\boldsymbol{B}, \tag{6.14}$$

  where a vector formula $\nabla\times(\nabla\times\boldsymbol{B})=\nabla(\nabla\cdot\boldsymbol{B})-\triangle\boldsymbol{B}$ and $\nabla\cdot\boldsymbol{B}=0$ are used. The first term of the right-hand side is the convection term and the second is the diffusion term. The coefficient $1/\mu_0\sigma$ in the second term has a dimension of the square of length per time, and is called the magnetic diffusivity.

  Denote the typical velocity by V and the typical length by L. Then the magnetic Reynolds number R_m is defined by $R_m=\mu_0\sigma VL$. The magnetic Reynolds number represents the ratio of the two terms in the right-hand side of Eq. (6.14). When $R_m\gg 1$, the diffusion term is negligible, and the dynamics of such a fluid is called the ideal magnetohydrodynamics (the ideal MHD).

- The energy equation for the magnetic field is easily derived from Eq. (6.3) rather than from Eq. (6.14). Taking the scalar product of $\boldsymbol{B}$ and Eq. (6.3) yields

$$\frac{1}{\mu_0}\boldsymbol{B}\cdot\frac{\partial \boldsymbol{B}}{\partial t}=-\frac{1}{\mu_0}\boldsymbol{B}\cdot(\nabla\times\boldsymbol{E}). \tag{6.15}$$

  The left hand-side is $\dfrac{\partial}{\partial t}\dfrac{|\boldsymbol{B}|^2}{2\mu_0}$. The right-hand side is transformed with a formula $\nabla\cdot(\boldsymbol{A}\times\boldsymbol{B})=\boldsymbol{B}\cdot(\nabla\times\boldsymbol{A})-\boldsymbol{A}\cdot(\nabla\times\boldsymbol{B})$ as

$$-\frac{1}{\mu_0}\boldsymbol{B}\cdot(\nabla\times\boldsymbol{E})=-\frac{1}{\mu_0}\left[\nabla\cdot(\boldsymbol{E}\times\boldsymbol{B})+\boldsymbol{E}\cdot(\nabla\times\boldsymbol{B})\right]. \tag{6.16}$$

  Furthermore, using Ohm's law and Ampère's law, we transform the second term on the right-hand side of this equation and obtain the energy equation for the magnetic field as

$$\frac{\partial}{\partial t}\left(\frac{|\boldsymbol{B}|^2}{2\mu_0}\right)=-\nabla\cdot\left(\boldsymbol{E}\times\frac{\boldsymbol{B}}{\mu_0}\right)-\boldsymbol{v}\cdot(\boldsymbol{j}_e\times\boldsymbol{B})-\frac{|\boldsymbol{j}_e|^2}{\sigma}. \tag{6.17}$$

  The left-hand side is the time derivative of the magnetic field energy per unit volume. The first term on the right side is the energy loss due to the energy transport of the Poynting vector, the second term represents the work done on the fluid by the Lorentz force, and the third term represents the magnetic energy lost as Joule heat.

Problem 33. Momentum conservation equation in MHD: The equation of motion (6.11) or (6.12) in MHD can be rewritten in the conservation form, as in the transformation of Euler's equation into Eq. (1.18). Derive the expression for the momentum flux density tensor, Π_{ij}, in MHD, which appears in the conservation form.

Problem 34. Derive Eq. (6.17).

Problem 35. In MHD, the energy of the electric field $\varepsilon_0 E^2/2$ is assumed to be much smaller than the magnetic field energy $B^2/2\mu_0$. The magnitude of the electric field $\boldsymbol{E}$ in MHD can be estimated from the first or second term of the right-hand side of Eq. (6.13). From the estimates from the first and second terms, show that the conditions for the electric field energy to be negligible are given by

$$v/c \ll 1, \qquad \mu_0 \sigma L c \gg 1, \tag{6.18}$$

respectively, where L is the length in which the magnetic field varies. Also, explain the physical meaning of the second condition.

6.2 Frozen-in and diffusion of the magnetic field

- Time evolution of the magnetic flux

  The magnetic flux across an area S, Φ_S, is given by

$$\Phi_S = \int_S \boldsymbol{B} \cdot \boldsymbol{n} dS. \tag{6.19}$$

When the area S moves together with the fluid, the time derivative of the magnetic flux Φ_S can be written as

$$\frac{d\Phi_S}{dt} = \int_S \frac{\partial \boldsymbol{B}}{\partial t} \cdot \boldsymbol{n} dS + \sum_k \boldsymbol{B}_k \cdot \frac{d\Delta \boldsymbol{S}_k}{dt}, \tag{6.20}$$

where the second term of the right-hand side represents the change due to the variation of S, and $\Delta \boldsymbol{S}_k$ is the ares of k-th small part of S multiplied by $\boldsymbol{n}$. The change in the area S is caused by the motion of the closed curve C, which is the boundary of S[30]. Denote the vector of a small interval which is a part of the closed curve C by $\Delta \boldsymbol{l}$. When the vector $\Delta \boldsymbol{l}$ moves with a velocity $\boldsymbol{v}$, the time derivative $d\Delta S/dt$ due to the motion of $\Delta \boldsymbol{l}$ is given by $\boldsymbol{v} \times \Delta \boldsymbol{l}$. Therefore, the second term is rewritten as

$$\sum_k \boldsymbol{B}_k \cdot \frac{d\Delta \boldsymbol{S}_k}{dt} = \sum_m \boldsymbol{B}_m \cdot (\boldsymbol{v}_m \times \Delta \boldsymbol{l}_m) = \oint_C (\boldsymbol{B} \times \boldsymbol{v}) \cdot d\boldsymbol{l} = \int_S [\nabla \times (\boldsymbol{B} \times \boldsymbol{v})] \cdot \boldsymbol{n} dS, \tag{6.21}$$

where the subscript m is the number assigned to each small part of the closed curve C. Furthermore, substituting the induction equation written as

$$\frac{\partial \boldsymbol{B}}{\partial t} = \nabla \times (\boldsymbol{v} \times \boldsymbol{B}) - \frac{1}{\sigma} \nabla \times \boldsymbol{j}_e, \tag{6.22}$$

[30]Since $\nabla \cdot \boldsymbol{B} = 0$, the magnetic flux is uniquely determined by the closed curve C.

into Eq. (6.20), we obtain

$$\frac{d\Phi_S}{dt} = -\frac{1}{\sigma}\int_S (\nabla\times\boldsymbol{j}_e)\cdot\boldsymbol{n}dS = -\frac{1}{\sigma}\oint_C \boldsymbol{j}_e\cdot d\boldsymbol{l}. \tag{6.23}$$

- **Frozen-in of the magnetic field**

  In a fluid with a sufficiently large conductivity σ, such as a fully ionized gas, the right-hand side of Eq. (6.23) is negligible. Such a case corresponds to the ideal MHD. In this case, the magnetic flux is constant for any area S moving with the fluid, and each magnetic field line must move with the fluid. This is the frozen-in of the magnetic flux (or magnetic field lines) into the fluid. Conversely, each fluid particle is frozen to a magnetic field line.

- **Diffusion of the magnetic field**

  In non-ideal MHD, the magnetic field diffuses according to the induction equation (6.14). The diffusion time is given by $L^2\mu_0\sigma$, where L is the length in which the magnetic field varies. The decrease (or increase) in the magnetic flux across S is because some magnetic field lines go out of (or enter) the area S, slipping from the fluid. Conversely, it is considered that the fluid slips from the magnetic field lines.

6.3 Alfv́en waves

Sound waves are waves due to the compressibility of fluids, but incompressible waves also exist in MHD. Let us investigate the latter MHD waves.

- As the unperturbed state, we consider a fluid of constant density ρ_0 at rest in a constant magnetic field $\boldsymbol{B}_0$. The temperature and pressure are also set to be constant.

- Perturbations: Denote the perturbations of the density, pressure, and velocity and the magnetic field by $\rho_1,\ p_1,\ \boldsymbol{v}_1,\ \boldsymbol{B}_1$, respectively. We also assume a perfect conductor ($\sigma\to\infty$) and adiabatic waves ($s_1 = 0$, $p_1 = c_s^2\,\rho_1$).

- Perturbation equations

Equation of continuity $\quad \nabla\cdot\boldsymbol{v}_1 = 0, \quad (6.24)$

$$\text{Euler's equation}\qquad \rho_0\frac{\partial\boldsymbol{v}_1}{\partial t} = -\nabla\left(c_s^2\rho_1 + \frac{1}{\mu_0}\boldsymbol{B}_0\cdot\boldsymbol{B}_1\right) + \frac{1}{\mu_0}(\boldsymbol{B}_0\cdot\nabla)\boldsymbol{B}_1, \tag{6.25}$$

$$\text{Induction equation}\qquad \frac{\partial\boldsymbol{B}_1}{\partial t} = \nabla\times(\boldsymbol{v}_1\times\boldsymbol{B}_0), \tag{6.26}$$

$$\nabla\cdot\boldsymbol{B}_1 = 0. \tag{6.27}$$

- Taking the divergence of Euler's equatoin (6.25) and using Eqs. (6.24) and (6.27), we have

$$\triangle\left(c_s^2\rho_1 + \frac{1}{\mu_0}\boldsymbol{B}_0\cdot\boldsymbol{B}_1\right) = 0. \tag{6.28}$$

Integrating this with the boundary condition of no divergence at far away yields

$$c_s^2\rho_1 + \frac{1}{\mu_0}\boldsymbol{B}_0 \cdot \boldsymbol{B}_1 = \text{const.} \tag{6.29}$$

Substituting this into Euler equation (6.25), we obtain

$$\frac{\partial \boldsymbol{v}_1}{\partial t} = \frac{1}{\rho_0\mu_0}(\boldsymbol{B}_0 \cdot \nabla)\boldsymbol{B}_1. \tag{6.30}$$

- The right-hand side of the induction equation (6.26) is rewritten using a vector formula, the equation of continuity, and $\boldsymbol{B}_0$ = constant, as

$$[\nabla\times(\boldsymbol{v}_1\times\boldsymbol{B}_0)]_i = \frac{\partial}{\partial x_l}(v_{1,i}B_{0,l}) - \frac{\partial}{\partial x_l}(v_{1,l}B_{0,i}) = [(\boldsymbol{B}_0 \cdot \nabla)\boldsymbol{v}_1]_i, \tag{6.31}$$

  and we obtain

$$\frac{\partial \boldsymbol{B}_1}{\partial t} = (\boldsymbol{B}_0 \cdot \nabla)\boldsymbol{v}_1. \tag{6.32}$$

- These perturbation equations have solutions in the form

$$\boldsymbol{v}_1 = \boldsymbol{v}_1' \exp(i\boldsymbol{k}\cdot\boldsymbol{x} - i\omega t), \qquad \boldsymbol{B}_1 = \boldsymbol{B}_1' \exp(i\boldsymbol{k}\cdot\boldsymbol{x} - i\omega t). \tag{6.33}$$

  Substituting these expressions into Eqs. (6.30) and (6.32), and eliminating $\boldsymbol{B}_1'$, we obtain the dispersion relation as

$$\omega^2 = \frac{1}{\rho_0\mu_0}(\boldsymbol{B}_0 \cdot \boldsymbol{k})^2. \tag{6.34}$$

  This wave is called the Alfvén wave. The group velocity of an Alfvén wave is given by

$$\frac{\partial \omega}{\partial \boldsymbol{k}} = \frac{\boldsymbol{B}_0}{\sqrt{\rho_0\mu_0}}. \tag{6.35}$$

  It is equal to the propagation velocity. The Alfvén waves propagate along the magnetic field line with the Alfvén velocity $v_A \equiv B_0/\sqrt{\rho_0\mu_0}$[31].

Consider the resilience in Alfvén waves. Suppose that a fluid moves in a magnetic field $\boldsymbol{B}_0$ with a velocity $\boldsymbol{v}$ perpendicular to $\boldsymbol{B}_0$. Then, a current density $\boldsymbol{j}_e$ in the direction of $\boldsymbol{v}\times\boldsymbol{B}_0$ is generated, and the Lorentz force $\boldsymbol{j}_e\times\boldsymbol{B}_0$ is exerted on the fluid. This Lorentz force has the opposite direction of $\boldsymbol{v}$ and acts as a resilience.

Another valid interpretation is that the resilience is caused by the magnetic tension. Waves in a string with a linear density λ under a force of tension T propagate with the velocity $\sqrt{T/\lambda}$. In the case of Alfvén waves, a magnetic field has the force of tension per unit area of B_0^2/μ_0, and the linear density is given by ρ_0. Thus the propagation velocity is obtained as $\sqrt{T/\lambda} = \sqrt{B_0^2/(\mu_0\rho_0)} = v_A$. That is, the propagation of the Alfvén waves is explained by the tension of the magnetic field lines.

[31] From §1.7 the applicable condition for the incompressible fluids is given by $\omega \ll kc_s$ or $v_A \ll c_s$.

6.4 Generalized Ohm's law and induction equation in weakly ionized plasma

Currents in magnetic fields are affected by the Hall effect and ambipolar diffusion under certain conditions. Therefore, it is necessary to modify Ohm's law of Eq. (6.6) for such cases. Here, we will investigate the microscopic motion of charged particles in the electromagnetic field in weakly ionized plasmas to derive the generalized Ohm's law. This generalization also modifies the induction equation in the magnetic field. The importance of this generalization in low-density, weakly ionized plasmas will be shown later.

In a weakly ionized plasma, charged particles (electrons, ions, and charged dust grains) are much fewer than neutral particles (atoms and molecules). Charged particles are accelerated by an electromagnetic field and decelerated by collisions primarily with neutral particles. Consider the average velocity $\boldsymbol{u}_i$ of the ith kind of charged particles of in the frame of reference where neutral particles are at rest. The equation of motion of a charged particle i with a mass m_i and a charge q_i can be written as

$$m_i \frac{d\boldsymbol{u}_i}{dt} = q_i(\boldsymbol{E} + \boldsymbol{u}_i \times \boldsymbol{B}) - \frac{m_i \boldsymbol{u}_i}{\tau_i}. \tag{6.36}$$

The second term on the right-hand side represents the drag force due to collisions with neutral particles, and τ_i is the deceleration time of the charged particle due to the collisions with neutral particles, which is inversely proportional to the number density of the neutral particles. For a light charged particle, the deceleration time τ_i is equal to the collision time with a neutral particle. However, when the charged particle is heavier than each neutral particle, τ_i is longer than the collision time by the mass ratio. In a weakly ionized plasma, collisions between charged particles can be ignored. The gravitational force is also negligible compared to the Lorentz force.

When the collision time is short enough, a steady state is realized where the two forces on the right side of Eq. (6.36) are balanced, and the acceleration term on the left side is negligible. We assume this steady state below. It is convenient to write this force balance in the magnetic field as

$$\boldsymbol{E} + \boldsymbol{u}_i \times \boldsymbol{B} - \frac{B}{\beta_i}\boldsymbol{u}_i = 0, \tag{6.37}$$

where the **Hall parameter** β_i is a dimensionless parameter given by the product of the epicycle frequency and the deceleration time of the charged particle. That is,

$$\beta_i = \frac{q_i B}{m_i}\tau_i. \tag{6.38}$$

Note that the sign of the Hall parameter depends on the sign of the charge.

We first consider the component of the current density parallel to the magnetic field. The velocity of a charged particle is divided into a component parallel to the magnetic field $\boldsymbol{u}_{i\parallel}$ and a component perpendicular to the magnetic field $\boldsymbol{u}_i'$. Dividing the electric field (and other vectors) into two components in the same way, we obtain the average velocity of charged particles i parallel to the magnetic field $\boldsymbol{u}_{i\parallel}$ from the parallel component of Eq. (6.37) as

$$\boldsymbol{u}_{i\parallel} = \frac{\beta_i}{B}\boldsymbol{E}_\parallel. \tag{6.39}$$

Then, the parallel component of the current density $\boldsymbol{j}_\parallel$ is given by

$$\boldsymbol{j}_\parallel = \sum_i q_i n_i \boldsymbol{u}_{i\parallel} = \sigma_{\rm C} \boldsymbol{E}_\parallel, \tag{6.40}$$

where the normal electric conductivity $\sigma_{\rm C}$ is given by

$$\sigma_{\rm C} = \frac{1}{B} \sum_i q_i n_i \beta_i. \tag{6.41}$$

Note that $\sigma_{\rm C}$ does not depend directly on B. Because electrons are less massive, their epicycle frequency is high, and $|\beta_e|$ is much larger than $|\beta_i|$ of other charged particles. Therefore, the contribution of electrons to the electrical conductivity is usually primary, and $\sigma_{\rm C} \simeq e n_e |\beta_e| / B$[32].

The component of the current density perpendicular to the magnetic field is obtained from the perpendicular component of Eq. (6.37)

$$\boldsymbol{E}' + \boldsymbol{u}_i' \times \boldsymbol{B} - \frac{B}{\beta_i} \boldsymbol{u}_i' = 0. \tag{6.42}$$

Taking the vector product of this equation and $\boldsymbol{B}$ and noting that $(\boldsymbol{u}_i' \times \boldsymbol{B}) \times \boldsymbol{B} = -\boldsymbol{u}_i' B^2$, we obtain

$$\boldsymbol{E}' \times \boldsymbol{B} - B^2 \boldsymbol{u}_i' - \frac{B}{\beta_i} (\boldsymbol{u}_i' \times \boldsymbol{B}) = 0. \tag{6.43}$$

Eliminating $\boldsymbol{u}_i' \times \boldsymbol{B}$ from Eqs. (6.42) and (6.43), we obtain the perpendicular component of the velocity $\boldsymbol{u}_i'$ as

$$\boldsymbol{u}_i' = \frac{\beta_i}{B(1+\beta_i)} \boldsymbol{E}' + \frac{\beta_i^2}{B^2(1+\beta_i)} \boldsymbol{E}' \times \boldsymbol{B}, \tag{6.44}$$

and the perpendicular component of the current density $\boldsymbol{j}'$ is can be written as

$$\boldsymbol{j}' = \sum_i q_i n_i \boldsymbol{u}_i' = \sigma_\perp \boldsymbol{E}' - \frac{\sigma_{\rm H}}{B} \boldsymbol{E}' \times \boldsymbol{B}, \tag{6.45}$$

where the coefficients $\sigma_\perp$ and $\sigma_{\rm H}$ are given by

$$\sigma_\perp = \frac{1}{B} \sum_i \frac{q_i n_i \beta_i}{1+\beta_i^2}, \qquad \sigma_{\rm H} = -\frac{1}{B} \sum_i \frac{q_i n_i \beta_i^2}{1+\beta_i^2} \tag{6.46}$$

respectively, and have the same dimension as $\sigma_{\rm C}$. To solve this equation for $\boldsymbol{E}'$, we perform the same operation as in the derivation of Eq. (6.44) from (6.42), and as a result, we obtain

$$\boldsymbol{E}' = \frac{1}{\sigma_\perp^2 + \sigma_{\rm H}^2} \left(\sigma_\perp \boldsymbol{j}' + \frac{\sigma_{\rm H}}{B} \boldsymbol{j}' \times \boldsymbol{B} \right). \tag{6.47}$$

[32] It can occur that the most of electrons are absorbed into dust grains. In such a limiting case, the electrical conductivity is determined by the terms of ions

Combining the parallel component of Eq. (6.40) and the perpendicular one of Eq. (6.47), and noting that $\boldsymbol{j} \times \boldsymbol{B} = \boldsymbol{j}' \times \boldsymbol{B}$ and $(\boldsymbol{j}' \times \boldsymbol{B}) \times \boldsymbol{B} = -\boldsymbol{j}' B^2$, we finally obtain the electric field as

$$\boldsymbol{E} = \frac{1}{\sigma_{\rm C}}\boldsymbol{j} + \frac{\eta_{\rm H}}{B}\boldsymbol{j} \times \boldsymbol{B} - \frac{\eta_{\rm A}}{B^2}(\boldsymbol{j} \times \boldsymbol{B}) \times \boldsymbol{B}, \tag{6.48}$$

where the new coefficients $\eta_{\rm H}$ and $\eta_{\rm A}$ are given by

$$\eta_{\rm H} = \frac{\sigma_{\rm H}}{\sigma_\perp^2 + \sigma_{\rm H}^2}, \qquad \eta_{\rm A} = \frac{\sigma_\perp}{\sigma_\perp^2 + \sigma_{\rm H}^2} - \frac{1}{\sigma_{\rm C}}, \tag{6.49}$$

respectively. Up to this point, we used the rest frame of neutral particles. This is the rest frame of the fluid since the neutral particles have the most of the fluid mass. The electric field in a frame of reference where the fluid flows with a velocity $\boldsymbol{v}$ is obtained as

$$\boldsymbol{E} = -\boldsymbol{v} \times \boldsymbol{B} + \frac{1}{\sigma_{\rm C}}\boldsymbol{j} + \frac{\eta_{\rm H}}{B}\boldsymbol{j} \times \boldsymbol{B} - \frac{\eta_{\rm A}}{B^2}(\boldsymbol{j} \times \boldsymbol{B}) \times \boldsymbol{B}. \tag{6.50}$$

This is the **generalized Ohm's law**. The third term on the right side represents the Hall effect, and the fourth is due to the ambipolar diffusion. Furthermore, by using Eq. (6.50), the generalized induction equation that includes the Hall effect and the ambipolar diffusion can be obtained as

$$\frac{\partial \boldsymbol{B}}{\partial t} = \nabla \times \left[\boldsymbol{v} \times \boldsymbol{B} - \frac{1}{\mu_0 \sigma_{\rm C}}\nabla \times \boldsymbol{B} - \frac{\eta_{\rm H}}{\mu_0 B}(\nabla \times \boldsymbol{B}) \times \boldsymbol{B} + \frac{\eta_{\rm A}}{\mu_0 B^2}((\nabla \times \boldsymbol{B}) \times \boldsymbol{B}) \times \boldsymbol{B}\right]. \tag{6.51}$$

Finally, let us see under which conditions the Hall effect (or the ambipolar diffusion) becomes effective. To do this, we estimate the magnitudes of the coefficients of the Hall effect and the ambipolar diffusion, $\eta_{\rm H}$, $\eta_{\rm A}$. These coefficients are determined by the abundances of each type of charged particles or by the Hall parameters β_i (see Eq. [6.46]). As mentioned above, the Hall parameters are inversely proportional to the number density (or density) of the neutral particles, and increase as the decreasing density. And $|\beta_e|$ of electrons is larger than $|\beta_i|$ of other charged particles. Therefore, as the gas density decreases, the Hall parameters vary from the following case A to case C: (A) $|\beta_i| \ll 1$ for all kinds of charge particles, (B) $|\beta_e| \gg 1$ and $|\beta_i| \ll 1$ for other charged particles, and (C) $|\beta_i| \gg 1$ for all kinds of charged particles. We examine the magnitudes of the coefficients for the Hall effect and the ambipolar diffusion for these three cases.

Case A: $|\beta_i| \ll 1$ for all kinds of charged particles

This corresponds to the case of high gas densities. In this case, since the denominators in Eq. (6.46) are approximately unity, $\sigma_\perp$ is equal to $\sigma_{\rm C}$, and they are much larger than $\sigma_{\rm H}$. Furthermore, these are determined by the electron terms since $|\beta_e| \gg |\beta_i|$. As a result, from Eq (6.49), we see that the coefficients $\eta_{\rm H}$ and $\eta_{\rm A}$ are both much smaller than $1/\sigma_{\rm C}$. Thus, the terms of the Hall effect and the ambipolar diffusion can be ignored compared to the ohmic dissipation term in the generalized Ohm's law (6.50) and the induction equation (6.51).

Case B: $|\beta_e| \gg 1$ **and** $|\beta_i| \ll 1$ **for other charged particles**

This case has a lower gas density than Case A but a higher one than Case C. In this case, $\sigma_{\rm H}$ is determined by the electron term and is approximately given by $\sigma_{\rm H} \simeq en_e/B \simeq \sigma_{\rm C}/|\beta_e|$. On the other hand, $\sigma_\perp$ is much smaller than $\sigma_{\rm H}$. Thus, we find that $\eta_{\rm H} \gg 1/\sigma_{\rm C}$, $\eta_{\rm A}$. Therefore, the Hall effect term is much larger than the ohmic dissipation and the ambipolar diffusion terms in the generalized Ohm's law and induction equation[33].

Case C: $|\beta_i| \gg 1$ **for all kinds of charged particles**

This corresponds to the case where the gas density is low enough (or the magnetic field is strong enough). In this case, the factors of β_i^2 of the numerator and denominator cancel out in each term of $\sigma_{\rm H}$, and $\sigma_{\rm H}$ becomes as small as $\mathrm{O}(|\beta_i|^{-2})$ due to the charge neutrality. On the other hand, $\sigma_\perp$ is $O(|\beta_i|^{-1})$. Then, the coefficients satisfy $\eta_{\rm A} \gg \eta_{\rm H} \gg 1/\sigma_{\rm C}$, and the ambipolar diffusion term overcomes the others. As a result, the effective electrical conductivity in the direction perpendicular to the magnetic field is extremely small. The low electrical conductivity is due to the epicycle motion of the charged particles (or the frozen-in of the charged particles to the magnetic field line). Then, the collisions with neutral particles help the electrical conduction in this direction. The conductivity $\sigma_\perp$ is determined by ions, which have a low epicycle frequency and a large gyration radius. Thus, $\sigma_\perp = q_i n_i/(\beta_i B) = m_i n_i/(\tau_i B^2)$.

[33]When $\eta_{\rm H}/B = 1/(en_e)$ is constant, the term of the Hall effect in the induction equation is perpendicular to $\boldsymbol{B}$, and cannot change the magnetic energy.

7 Mechanics of Disk Objects

7.1 Basic equations

(a) Astronomical disk objects

An astronomical object that contracts due to the self-gravity will inevitably rotate at high speed due to the reduction of its moment of inertia if its angular momentum is not efficiently transferred to the outside due to a magnetic field or other process. When the rotational velocity increases to the point where centrifugal force and gravity are roughly balanced, the object becomes disk-shaped. Examples of such disk objects are as follows.

- Galaxy = bulge + galactic disk + halo.
- Accretion disks, e.g., black-hole accretion disks, protoplanetary disks, etc.

(b) Basic equations for two-dimensional disks

- Two-dimensional approximation:
  Thin disks are assumed. Their vertical structure will be described later.

$$\text{Surface density:} \quad \Sigma = \int_{-\infty}^{\infty} \rho dz, \qquad \text{2D pressure:} \quad P = \int_{-\infty}^{\infty} p dz, \tag{7.1}$$

$$v_z = 0, \quad \frac{\partial \boldsymbol{v}}{\partial z} = 0. \tag{7.2}$$

- Equation of continuity (2D polar coordinates (r, ϕ))

$$\frac{\partial \Sigma}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}(r\Sigma v_r) + \frac{1}{r}\frac{\partial}{\partial \phi}(\Sigma v_\phi) = 0. \tag{7.3}$$

- Two-dimensional Euler's equation[34]

$$r \text{ component:} \quad \frac{\partial v_r}{\partial t} + v_r\frac{\partial v_r}{\partial r} + \frac{v_\phi}{r}\frac{\partial v_r}{\partial \phi} - \frac{v_\phi^2}{r} = -\frac{1}{\Sigma}\frac{\partial P}{\partial r} - \frac{GM_c}{r^2} - \frac{\partial \Phi_{\rm D}}{\partial r}, \tag{7.4}$$

$$\phi \text{ component:} \quad \frac{\partial v_\phi}{\partial t} + v_r\frac{\partial v_\phi}{\partial r} + \frac{v_\phi}{r}\frac{\partial v_\phi}{\partial \phi} + \frac{v_r v_\phi}{r} = -\frac{1}{r\Sigma}\frac{\partial P}{\partial \phi} - \frac{1}{r}\frac{\partial \Phi_{\rm D}}{\partial \phi}, \tag{7.5}$$

  where M_c is the mass of the central object and $\Phi_{\rm D}$ is the gravitational potential of the disk.

- 3D Poisson's equation

$$\triangle \Phi_{\rm D} = 4\pi G \Sigma \delta(z) \tag{7.6}$$

[34] In 2D polar coordinates, additional terms $-\boldsymbol{e}_r v_\phi^2/r + \boldsymbol{e}_\phi v_r v_\phi/r$ appear in $(\boldsymbol{v} \cdot \mathbf{grad})\boldsymbol{v}$. The velocity vector is expressed with basis vectors as $\boldsymbol{v} = v_r \boldsymbol{e}_r + v_\phi \boldsymbol{e}_\phi$. Noting that $\mathbf{grad} = (\partial/\partial r, 1/r\partial/\partial\phi)$ also acts on the basis vectors, and using $\partial \boldsymbol{e}_r/\partial\phi = \boldsymbol{e}_\phi$ and $\partial \boldsymbol{e}_\phi/\partial\phi = -\boldsymbol{e}_r$, we can obtain these additional terms.

- Equation of state

$$P = K'\Sigma^{\gamma'} \qquad (\gamma' = 1 \text{ for isothermal case}). \tag{7.7}$$

Problem 36. Derive the equation of angular momentum conservation that takes the gravity the central object into account, from the equation of continuity (7.3) and the ϕ component of Euler's equation (7.5). Disk gravity can be treated as an external force.

(c) Vertical structure of disks

- For a thin disk, the vertical hydrostatic equation is obtained from the z component of Euler's equation as

$$\frac{1}{\rho}\frac{\partial p}{\partial z} = -\Omega'^2 z, \tag{7.8}$$

where the vertical angular frequency Ω' is given by

$$\Omega'^2 = \frac{GM_c}{r^3} + \frac{\partial^2 \Phi_{\mathrm{D}}}{\partial z^2}(z = 0). \tag{7.9}$$

If the disk gravity is negligible, Ω' is equal to the Keplerian angular velocity $(GM_c/r^3)^{1/2}$.

- The vertical density profile of the disk can be obtained by solving the vertical hydrostatic equation (7.8). For vertically isothermal cases, we have

$$\rho(z) = \frac{\Sigma}{\sqrt{2\pi}h}\, e^{-z^2/2h^2}. \tag{7.10}$$

The vertical disk scale height h is given by

$$h = \frac{c_s}{\Omega'}, \tag{7.11}$$

where c_s is the isothermal sound velocity with $\gamma = 1$. For polytropic disks, we obtain

$$\rho(z) = \rho(0)\left(1 - \frac{(\gamma - 1)z^2}{2h^2}\right)^{1/(\gamma-1)}. \tag{7.12}$$

In this case, h is also given by Eq. (7.11), but c_s in it is evaluated at $z = 0$.

- Flows obtained using the two-dimensional disk approximation is valid if the characteristic length of flows is sufficiently longer than the disk scale height h. However, for local phenomena in disks, the characteristic length is often comparable to h. In such cases, 2D approximation is not accurate, but the error is expected to be of the order of unity and the qualitative properties of the flows would not change. We can say that the two-dimensional disk approximation is not valid for phenomena where the characteristic length is much shorter than h.

Problem 37. Derive the equations of the vertical density profile (7.10) and (7.12).

Problem 38. If the gas disk satisfies the polytropic relation (1.25) with $\gamma > 1$, show that a similar power relation (7.7) holds between the surface density Σ and the two-dimensional pressure P. This relation is obtained by integrating the disk vertical profile (7.12). Also find the relation between the indexes γ and γ'.
(Hint: First, find the power-law dependences of Σ and P on $\rho(0)$.)

7.2 Gravitational instability of disks

Using the basic equations (7.3)-(7.7) of the two-dimensional approximation in the previous section, we can conduct a linear stability analysis and investigate the self-gravitational instability of disks. Here, we particularly examine local gravitational instability.

(a) Perturbations

Each quantity is divided into an unperturbed component (index 0) and a perturbed component (index 1).

$$\Sigma = \Sigma_0 + \Sigma_1, \quad P = P_0 + P_1 \quad (P_1 = c_s^2 \Sigma_1),$$
$$v_r = v_{r,1}, \quad v_\phi = r\Omega(r) + v_{\phi,1}, \quad \Phi_{\rm D} = \Phi_{\rm D,0} + \Phi_{\rm D,1},$$

where $c_s^2 = \gamma' K' {\Sigma_0}^{\gamma'-1}$. From the balance among the centrifugal force, the gravitational forces by the central star and the disk, and the pressure gradient, the angular velocity, Ω, of the unperturbed disk rotation is obtained as

$$\Omega^2 = \frac{GM_c}{r^3} + \frac{1}{r}\frac{\partial \Phi_{\rm D,0}}{\partial r}(z=0) + \frac{1}{r}\frac{\partial H_0}{\partial r}, \tag{7.13}$$

where $H_0 = \frac{\gamma'}{\gamma'-1} P_0/\Sigma_0$ is the unperturbed enthalpy. The angular velocity Ω generally has a radial dependence. In the Keplerian rotation, $\Omega \propto r^{-3/2}$ while $\Omega \propto 1/r$ for galactic disk. A rotation in which Ω depends on r is called a differential rotation, and a rotation in which Ω is independent of r is called a rigid rotation.

(b) WKB approximation

We consider perturbations with sufficiently large wave numbers in the radial direction. For such perturbations, the unperturbed state can be regarded as constant at a distance of the wavelength, so

$$\text{perturbations} \propto \exp(ikr + im\phi - i\omega t), \tag{7.14}$$

and the derivative of them satisfies

$$\left|\frac{\partial}{\partial r}\right| \gg \frac{1}{r}\left|\frac{\partial}{\partial \phi}\right|, \ \frac{1}{r}. \tag{7.15}$$

This is the WKB (Wentzel–Kramers–Brillouin) approximation. It assumes that the radial component of the wavenumber vector is much larger than that the ϕ component. This assumption is reasonable because the perturbations are stretched in the ϕ direction in a differentially rotating disk.

(c) Vertical integration of Poisson's equation

Under the WKB approximation, the perturbation equation of Poisson's equation (7.6) is written as

$$\left(\frac{\partial^2}{\partial r^2} + \frac{\partial^2}{\partial z^2}\right) \Phi_{\mathrm{D},1} = 4\pi G \Sigma_1 \delta(z). \tag{7.16}$$

Integrating this equation from $z = -\epsilon$ to $+\epsilon$ ($\epsilon \ll 1$), and assuming vertical symmetric disks, we obtain

$$\left(\frac{\partial \Phi_{\mathrm{D},1}}{\partial z}\right)_{z=+0} = -\left(\frac{\partial \Phi_{\mathrm{D},1}}{\partial z}\right)_{z=-0} = 2\pi G \Sigma_1. \tag{7.17}$$

Furthermore, since Eq. (7.16) is given by $\left(\frac{\partial^2}{\partial r^2} + \frac{\partial^2}{\partial z^2}\right) \Phi_{\mathrm{D},1} = 0$ for $z \neq 0$, we have

$$\Phi_{\mathrm{D},1} \propto e^{-k|z|} \exp(ikr + im\phi - i\omega t). \tag{7.18}$$

From Eqs. (7.17) and (7.18), we obtain

$$\Phi_{\mathrm{D},1}(z = 0) = -2\pi G \Sigma_1 / k. \tag{7.19}$$

(d) Other perturbation equations and a dispersion relation

- Equation of continuity

$$i(m\Omega - \omega)\Sigma_1 + ik\Sigma_0 v_{r,1} + \frac{im\Sigma_0}{r} v_{\phi,1} = 0. \tag{7.20}$$

- Euler's equation

$$\begin{pmatrix} i(m\Omega - \omega) & -2\Omega \\ -2B & i(m\Omega - \omega) \end{pmatrix} \begin{pmatrix} v_{r,1} \\ v_{\phi,1} \end{pmatrix} = \left(\frac{c_s^2 \Sigma_1}{\Sigma_0} + \Phi_{D,1}\right) \begin{pmatrix} -ik \\ -im/r \end{pmatrix}. \tag{7.21}$$

Solving this yields

$$\begin{pmatrix} v_{r,1} \\ v_{\phi,1} \end{pmatrix} = \frac{1}{\Delta} \left(\frac{c_s^2 \Sigma_1}{\Sigma_0} + \Phi_{D,1}\right) \begin{pmatrix} (m\Omega - \omega)k \\ -i2Bk \end{pmatrix}, \tag{7.22}$$

where

$$\begin{aligned} B &= -\frac{1}{2r}\frac{d(r^2\Omega)}{dr} && \text{(Oort constant)}, \\ \Delta &= \kappa^2 - (m\Omega - \omega)^2, \\ \kappa^2 &= -4B\Omega && \text{(epicycle frequency)}. \end{aligned} \tag{7.23}$$

- Substituting Eqs. (7.19) and (7.22) into (7.20), we have

$$\left(1+\frac{c_s^2k^2-2\pi G\Sigma_0 k}{\Delta}\right)\Sigma_1=0. \tag{7.24}$$

  Therefore, we obtain the dispersion relation

$$(m\Omega-\omega)^2=c_s^2k^2-2\pi G\Sigma_0 k+\kappa^2. \tag{7.25}$$

(e) Stability condition

- If the frequency ω is real, the perturbation is stable. For real ω, the right-hand side of the dispersion relation must be positive for all k. That is, the stability condition is that "$\kappa^2>0$ " and "the discriminant for the right-hand side = 0 is negative" hold. The former requires that the specific angular momentum l $(=r^2\Omega)$ increases with r. It is called the Rayleigh's stability condition for rotating disks. From the latter, we obtain Toomre's stability condition

$$Q\equiv\frac{c_s\kappa}{\pi G\Sigma}>1. \tag{7.26}$$

  where Q is called **Toomre's Q value**.

- The critical wavelength with $Q=1$ is given by

$$\lambda_{\rm crit}=2\pi/k_{\rm crit}=2\pi c_s/\kappa. \tag{7.27}$$

  Since three frequencies κ,Ω,Ω' are comparable, the critical wavelength is comparable with the disk scale height h. Therefore, Toomre's stability condition may change somewhat due to the effect of the disk thickness. On the other hand, for a thin disk, $\lambda_{\rm crit}$ is sufficiently short compared to the disk radius, so the WKB approximation is accurate.

- With the above analysis, it is not possible to clarify the instability of global perturbations whose wavelength is comparable with the disk radius. Studies on the excitation of such global modes due to gravitational instability show that a two-armed spiral wave is excited at $Q\sim1$. In other words, Toomre's Q value is a good index of gravitational instability even for global modes.

Problem 39. Q value of the galactic disk at the solar neighborhood.
The combined density of stars and interstellar gas near the sun, at 8 kpc from the galactic center, is estimated to be about 0.1 $M_\odot/{\rm pc}^3$. The surface density Σ of the galactic disk can be estimated by the product of this local density and the disk thickness h. Assuming that the speed of disk rotation is 200 km/s and approximating $\Omega'=\kappa$, estimate the Q value of the galactic disk at the solar neighborhood. (The solar mass is 2×10^{30} kg and $1{\rm pc}=3\times10^{16}$m.) The result of the estimation is $Q\simeq1$. This suggests that the spiral structure of the galactic disk is due to gravitational instability.

Problem 40. Q value of a protoplanetary disk. The temperature of a protoplanetary disk is determined by the radiation from the host star, and is approximately given by

$$T = \left(\frac{L_\odot}{16\pi\sigma r^2}\right) \simeq 300(r/1\mathrm{AU})^{-1/2}\mathrm{K} \tag{7.28}$$

for the protoplanetary disk around the sun (Hayashi et al. 1986). Furthermore, the surface density of the disk that created the solar system is expected to be 2000 g/cm^2 at 1 AU. Find the sound velocity at 1 AU and estimate the Q value of the protoplanetary disk at 1 AU. Also, find the disk surface density at 100 AU, $\Sigma_{100\mathrm{AU}}$, when the disk has $Q = 1$ at 100AU (a typical disk radius), and estimate the mass of a marginally unstable disk using $M_d = \pi(100\mathrm{AU})^2\Sigma_{100\mathrm{AU}}$.

7.3 Evolution and structure of accretion disks

Accretion disks around black holes and protoplanetary disks evolve due to viscosity. Here we describe the evolution of viscous accretion disks. An accretion disk rotates around its host star at approximately Keplerian angular velocity. It is a differential rotation, where the inner part rotates rapidly and the outer part rotates slowly. When viscosity acts on a differentially rotating disk, the rapidly rotating inner disk material experiences a negative torque from the outer material and slows down. Thus, viscosity transfers angular momentum from the inner disk to the outer disk. The inner disk loses the angular momentum and falls inward, while the outer disk expands outward. This results in mass accretion onto the host star, reducing the disk mass and increasing the disk radius.

7.3.1 Basic equations for accretion disks

We examine the evolution of the viscous accretion disk in detail using the hydrodynamical equations with the addition of viscous effects. We consider an axi-symmetric disk, which is assumed here to be a two-dimensional disk. To describe the disk, we use the polar coordinate system (R, ϕ) with the host star at the origin. For two-dimensional axi-symmetric disks, the equation of continuity (7.3) is rewritten as

$$\frac{\partial \Sigma}{\partial t} + \frac{1}{R}\frac{\partial}{\partial R}\left(R\Sigma v_R\right) = 0. \tag{7.29}$$

We do not consider any inflow onto the disk or outflow except the accretion onto the host star. For accretion disks, we use the Navier-Stokes equation with the viscosity term instead of the Euler equation. Around a host star, the Navier-Stokes equation is given by

$$\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v}\cdot\mathbf{grad})\boldsymbol{v} = -\frac{1}{\rho}\mathbf{grad}\, p + \mathbf{grad}\left(\frac{GM_c}{r}\right) + \frac{1}{\rho}\mathrm{div}\,\boldsymbol{\Pi'} \tag{7.30}$$

where Π' is the viscous stress tensor given by[35]

$$\Pi'_{ij} = \rho\nu\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right) \tag{7.31}$$

[35]For accretion disks, the gas velocity in the frame rotating with the disk is smaller than the sound speed and we can assume the incompressible fluid.

where ν is the kinetic viscosity. We do not consider any external forces other than the gravity of the host star. For two-dimensional axi-symmetric disks, the ϕ-component of the Navier-Stokes equation is rewritten as[36]

$$\frac{\partial v_\phi}{\partial t} + v_r \frac{\partial v_\phi}{\partial r} + \frac{v_r v_\phi}{r} = \frac{1}{\Sigma}\left(\frac{1}{r}\frac{\partial\, r\Pi'_{r\phi}}{\partial r} + \frac{\Pi'_{r\phi}}{r}\right) \tag{7.32}$$

and the r,ϕ-component of the viscous stress, $\Pi'_{r\phi}$, is given for two-dimensional disks by

$$\Pi'_{r\phi} = \Sigma\nu\left(\frac{\partial v_\phi}{\partial r} - \frac{v_\phi}{r}\right) = \Sigma\nu r\frac{d\Omega}{dr}. \tag{7.33}$$

By the definition of the viscous stress, $\Pi'_{r\phi}(r)$ represents the ϕ component of the force (per unit length) exerted on the inner disk material by the outer material tangent to the inner material at R. The angular velocity of the disk, Ω, is determined by the balance mainly between the stellar gravity and centrifugal force in the radial component of the equation (7.30) and is approximately given by $\Omega_{\rm K}$. As seen in the equation (7.33), the sign of the viscous stress $\Pi'_{R\phi}$ is determined by the gradient of Ω. In a uniformly rotating disk with constant Ω, the viscous stress does not work. In Keplerian disks with $\Omega_{\rm K}$, the negative viscous torque is exerted on the inner material by the outer material.

The angular momentum conservation equation for accretion disks is obtained from the equations (7.29), (7.32), and (7.33) as

$$\frac{\partial}{\partial t}(\Sigma\, j) + \frac{1}{r}\frac{\partial}{\partial r}\left(r\Sigma\, j v_r - r^3\Sigma\nu\frac{d\Omega}{dr}\right) = 0, \tag{7.34}$$

where $j\,(=R^2\Omega)$ is the specific angular momentum. The second term in the left-hand side of the equation (7.34) is the divergence of the radial angular momentum flux (density). The first term of the angular momentum flux shows the flux due to advection and the second is that due to the viscous torque. Using the equations (7.29) and (7.34), and noting that $\partial j/\partial t = 0$, we also obtain v_R and the inward mass flux (i.e., the accretion rate) of the disk as

$$\dot{\mathcal{M}} \equiv -2\pi r\Sigma\, v_r = -\frac{2\pi}{(dj/dr)}\frac{\partial}{\partial r}\left(r^3\Sigma\,\nu\frac{d\Omega}{dr}\right). \tag{7.35}$$

Substituting this into the equation (7.29), we finally obtain the equation describing the viscous evolution of accretion disks as

$$\frac{\partial\Sigma}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}\left[\frac{1}{(dj/dr)}\frac{\partial}{\partial r}\left(r^3\Sigma\nu\frac{d\Omega}{dr}\right)\right] = 0. \tag{7.36}$$

[36] In the polar coordinate system, the term of $(\boldsymbol{v}\cdot\mathbf{grad})\boldsymbol{v}$ has additional terms $-\boldsymbol{e}_R v_\phi^2/R + \boldsymbol{e}_\phi v_R v_\phi/R$ and the latter one appears in the equation (7.32). Since the velocity vector is expressed as $\boldsymbol{v} = v_R\boldsymbol{e}_R + v_\phi\boldsymbol{e}_\phi$, the gradient operates on the basis vectors as well as the velocity components. Noting this, and using $\partial\boldsymbol{e}_R/\partial\phi = \boldsymbol{e}_\phi$ and $\partial\boldsymbol{e}_\phi/\partial\phi = -\boldsymbol{e}_R$, we can obtain the additional terms above. The additional term of the viscosity term in the equation (7.32) and that of the viscous stress in the equation (7.33) are also derived in the same way.

For a disk in Keplerian rotation with $\Omega = \Omega_K$, it is reduced to

$$\frac{\partial \Sigma}{\partial t} + \frac{3}{r}\frac{\partial}{\partial r}\left[r^{1/2}\frac{\partial}{\partial r}\left(r^{1/2}\nu\Sigma\right)\right] = 0. \tag{7.37}$$

The viscosity of accretion disks is determined by the turbulent viscosity, not the molecular viscosity. The origin of the turbulence has not yet been determined for protoplanetary disks, although magneto-rotational instability and self-gravitational instability are strong candidates, and the magnitude of the viscosity has a large uncertainty. Therefore the kinetic viscosity of accretion disks is often expressed in terms of a non-dimensional parameter as

$$\nu = \alpha h^2 \Omega. \tag{7.38}$$

This simple expression for the viscosity is known as the Shakura-Sunyaev α prescription and α is called Shakura-Sunyaev α parameter (Shakura & Sunyaev, 1973).

7.3.2 Solution for steady-state accretion disks

We consider a steady solution for (inward) accreting disks. Setting $\partial/\partial t = 0$ in the equations (7.36) and (7.34), we obtain the mass and angular momentum fluxes as

$$\dot{\mathcal{M}} = \text{const.}, \tag{7.39}$$

$$\dot{\mathcal{J}} \equiv j\dot{\mathcal{M}} + 2\pi r^3 \Sigma \nu \frac{d\Omega}{dr} = \text{const.} \tag{7.40}$$

Note that inward fluxes are defined to be positive. For accretion disks, thus, $\dot{\mathcal{M}}$ is positive. Furthermore, assuming $\Sigma = 0$ at the inner disk edge $r_{\rm in}$ as the inner boundary condition, we obtain

$$\dot{\mathcal{J}} = \dot{\mathcal{M}} j(r_{\rm in}). \tag{7.41}$$

Substituting this into the equation (7.40), we obtain the steady surface density as

$$\Sigma = -\dot{\mathcal{M}} \frac{j(r) - j(r_{\rm in})}{2\pi\nu r^3 (d\Omega/dr)}. \tag{7.42}$$

For Keplerian disks, it is rewritten as

$$\Sigma = \frac{\dot{\mathcal{M}}}{3\pi\nu}\left(1 - \sqrt{\frac{r_{\rm in}}{r}}\right). \tag{7.43}$$

7.3.3 Similarity solution for accretion disks

Next we consider a time-evolving solution to equation (7.37). Suppose that the kinematic viscosity is given by a power-law function

$$\nu = \nu_0 R^{\gamma}. \tag{7.44}$$

In this case, it is known that there exists a similarity solution (Lynden-Bell & Pringle 1974)[37]. In the similarity solution for an accretion disk, the disk radius and surface density evolve with time, but the surface density distribution remains in a similar form.

(a) Dimensional analysis for evolution of accretion disks

The time evolution of the similarity solution can be clarified by dimensional analysis. An accretion disk spreads due to viscosity and its radius R_d increases. Suppose that the disk is formed at $t = 0$ in a small size and then spreads out due to the viscous effect. Since the equation (7.37) is a second-order differential equation for space and has the form of a diffusion equation, time t is approximately equal to the viscous diffusion time of the disk, $R_d^2/\nu(R_d)$. Then, also using the equation (7.44), the radius of the disk is approximately given by a power-law function of time as

$$R_d \simeq (\nu_0 t)^{\frac{1}{2-\gamma}}. \tag{7.45}$$

Note that ν_0 does not have the dimension of the diffusion coefficient [cm^{2}s^{-1}].

In the similarity solution, the inner edge radius R_in is assumed to be much smaller than the disk radius. Since the angular momentum flux at the inner edge is also negligibly small, the total angular momentum, J_d, of the disk is conserved. We can use the constant J_d to estimate the evolution of the disk mass. Estimating the characteristic value of the specific angular momentum of the disk as $R_d^2\,\Omega(R_d)$, the disk mass M_d is approximately given by

$$M_\mathrm{d} \simeq \frac{J_\mathrm{d}}{R_d^2\,\Omega(R_d)} \quad \propto t^{-\frac{1}{2(2-\gamma)}}. \tag{7.46}$$

In the above, the time dependence is derived using $\Omega \propto R^{-3/2}$. Furthermore, the characteristic value of the disk surface density can be estimated as

$$\Sigma(R_\mathrm{d}(t), t) \simeq \frac{J_\mathrm{d}}{R_d^4\,\Omega(R_d)} \quad \propto t^{-\frac{5}{2(2-\gamma)}}. \tag{7.47}$$

The similarity solution for the surface density also depend on a non-dimensional "similarity" variable, $y = r^{2-\gamma}/(\nu_0 t) \simeq (r/R_\mathrm{d})^{2-\gamma}$. The surface density distribution of the disk is determined by its y-dependence.

(b) Exact form of the similarity solution

The similarity solution to (7.37) is written as (Lynden-Bell & Pringle 1974; Hartmann et al. 1998; see also Appendix for the derivation)

$$\Sigma(r,t) = \frac{|\dot{M}_\mathrm{d}(t)|}{3\pi\nu} \exp\left[-\left(\frac{r}{R_d(t)}\right)^{2-\gamma}\right], \tag{7.48}$$

The disk radius R_d is given by

$$R_d = \left[3(2-\gamma)^2 \nu_0 t\right]^{\frac{1}{2-\gamma}} \tag{7.49}$$

[37]Surprisingly, a similarity solution also exists in a non-linear case where the kinematic viscosity is given by $\nu = \nu_0 R^\gamma \Sigma^\delta$ (Pringle 1974; Cannizzo et al. 1990). Even in the non-linear case, the time evolution of the disk radius and mass can be estimated by dimensional analysis similar to (a) in this subsection and the exact solution can be obtained from a derivation similar to (b).

and the disk mass $M_{\rm d}$ and its time derivative are

$$M_{\rm d} = \frac{J_{\rm d}}{\Gamma(b)\, R_{\rm d}^2 \Omega(R_{\rm d})}, \qquad \dot{M}_{\rm d} = -\frac{M_{\rm d}}{2(2-\gamma)t}, \tag{7.50}$$

where $b = (5-2\gamma)/(4-2\gamma)$ and $\Gamma(b)$ is the Gamma function. We can see that these expressions of the similarity solution are consistent with the above estimates by dimensional analysis. We also find that the similarity solution (7.48) agrees with the steady solution (7.42) in the radial range of $r_{\rm in} \ll r \ll R_{\rm d}$. In this range Σ is proportional to $1/\nu$ or $r^{-\gamma}$, and it is exponentially truncated at a radius $R_{\rm d}$. The inward mass and angular momentum fluxes are written as

$$\dot{\mathcal{M}} = -2\pi r \Sigma v_r = 3\pi\nu\Sigma \left[1 - 2(2-\gamma)\left(\frac{r}{R_d}\right)^{2-\gamma}\right], \tag{7.51}$$

$$\dot{\mathcal{J}} = -6\pi(2-\gamma) j \nu\Sigma \left(\frac{r}{R_d}\right)^{2-\gamma}, \tag{7.52}$$

respectively. The angular momentum is always transferred outward in the similarity solution. The equation (7.51) also gives the radial velocity.

We estimate the life time of protoplanetary disks using the similarity solution. Adapting the α prescription for the viscosity (equation [7.38]) and assuming a constant α and $T \propto r^{1/2}$, we obtain $\nu \propto r$ and $\gamma = 1$. The disk life time is approximately given by

$$t_{\rm d} \simeq \frac{R_{\rm d}^2}{3(2-\gamma)^2\nu} = \frac{R_{\rm d}^2}{3\alpha h^2 \Omega(R_{\rm d})}. \tag{7.53}$$

If $\alpha = 10^{-3}$, the life time of a protoplanetary disk is estimated to be 5 Myr for the disk with the radius of 100au (and $h/R_d \simeq 0.1$), which is almost consistent with the observed life time of protoplanetary disks. Thus we expect that $\alpha = 10^{-3}$ might be the typical value for protoplanetary disks. The second equation of (7.50) gives a simple relation between the mass accretion rate and the disk mass. For a 1Myr old disk with the mass of $0.02{\rm M}_\odot$, the mass accretion rate is obtained as $10^{-8}{\rm M}_\odot/{\rm yr}$, which is the typical value of the observed accretion rate.

(c) Derivation of the similarity solution

We briefly describe the derivation of the similarity solution. From the given parameters, ν_0 and J_d, and two independent variables, r and t, we can form only one dimensionless variable, which can be written as

$$y = \frac{r^2}{\nu t} = \frac{r^{2-\gamma}}{\nu_0 t}. \tag{7.54}$$

The time and radial dependences of the similarity solution are described only by this dimensionless variable y. The radial distribution of the angular momentum spreads out with the increase in the disk radius, but its distribution in the y-space does not change due to the similarity. That is, the angular momentum of the disk inside a radius r that changes so that $y =$constant, must remain constant. It is easy to write down the equation that expresses the constancy of this angular momentum. The angular momentum flowing out of a radius r in unit time is given by the angular momentum flux, $-\dot{\mathcal{J}}$, which is defined by the equation (7.40). On the other hand, since the radius r corresponding to a given y increases with the velocity of $\frac{dr(y)}{dt}$, an area inside the radius $r(y)$ increases in unit time by $2\pi r\frac{dr(y)}{dt}$, and the angular momentum in this additional area is $j\Sigma\, 2\pi r\frac{dr(y)}{dt}$. These angular momenta equals due to constancy of the angular momentum distribution in the y-space, and we obtain

$$\dot{\mathcal{M}} \equiv -2\pi r\Sigma v_r = 2\pi\beta\nu\Sigma\left[1 - \frac{y}{\beta(2-\gamma)}\right]. \tag{7.55}$$

Using Eqs. (7.49) and (7.54), we can see that this is equal to Eq. (7.51).

The solution of the surface density can be expressed as

$$\Sigma = \frac{J_d}{r^4\Omega(r)}f(y), \tag{7.56}$$

where $f(y)$ is a dimensionless function and the prefactor has the dimension of a surface density. Substituting these expressions for Σ and v_r into the equation (7.35), we obtain a differential equation for f as $d\ln f/d\ln y = -y/[3(2-\gamma)^2]+b$. Solving this equation yields the solution

$$f = \frac{2-\gamma}{2\pi\Gamma(b)}x^b\exp(-x) \tag{7.57}$$

with the new variable $x = y/[3(2-\gamma)^2]$, and also gives Σ. In the equation (7.57), the coefficient is determined by the condition that the total angular momentum calculated with Σ should equal J_d. Using this solution of Σ, we obtain the disk mass as the equation (7.50). Finally, using the equation (7.50), the solution of Σ is rewritten as the equation (7.48).

Problem 41. Show that $dr(y)/dt = \nu y/[(2-\gamma)r]$ and derive Eq. (7.55). Also derive the differential equation for $f(y)$ and its solution (7.57).

Problem 42. Derive Eqs. (7.50) and (7.48).

Problem 43. The obtained similarity solution (7.48) is physically meaningless when the power-law index γ of the viscosity is larger than 2. Find the physical reason why $\gamma < 2$ is required for the similarity solution by explaining how the physical property of the disk evolution changes between the cases with $\gamma < 2$ and $\gamma > 2$.

Problem 44. A similarity solution had also been derived for accretion disks where the viscosity also depends on the surface density as $\nu = \nu(r,\Sigma) = \nu_0 r^\gamma\Sigma^\delta$ (Pringle 1974, 1991; Cannizzo et al. 1990). Let us obtain such a similarity solution in the same way as above.

1. Using the characteristic disk radius R_c, the characteristic value of the surface density Σ_c is given by $\Sigma_c = J_d/[R_c^4\Omega(R_c)]$. The characteristic disk radius R_c also satisfies $R_c^2 = t\nu(R_c,\Sigma_c)$. Then, show $R_c(t) = [(J_d/\sqrt{GM_*})^\delta\nu_0 t]^{1/a}$, where $a = 2-\gamma+\frac{5}{2}\delta$ (cf. eq. [7.45]).

2. In this case, the dimensionless variable y is defined by $y \equiv r^2/[t\nu(r,\Sigma_r)] = [r/R_c(t)]^a$, where $\Sigma_r = J_d/[r^4\Omega(r)]$ (cf. eq. [7.54]). Solving this difinition of y for r yields $r(y) = R_c(t)y^{1/a}$. Then, show that $dr(y)/dt$ and the inward mass flux are given by

$$\frac{dr(y)}{dt} = \frac{\nu(r,\Sigma_r)y}{ar}, \qquad \dot{\mathcal{M}} = 3\pi\nu\Sigma\left[1 - \frac{2y}{3a}\left(\frac{\Sigma}{\Sigma_r}\right)^{-\delta}\right]. \tag{7.58}$$

The derivation is similar to that of Eq. (7.55).

3. The similarity solution is written as $\Sigma = \Sigma_r f(y)$. From the equivalence of Eqs. (7.35) and (7.58), derive the differential equation for $f(y)$, $\frac{df^\delta}{dy} - A\frac{f^\delta}{y} + B = 0$, where $A = \frac{1+\frac{1}{2a}}{1+\frac{1}{\delta}}$ and $B = \frac{1}{3a^2(1+\frac{1}{\delta})}$.

4. Solve the differential equation for $f(y)$ and show that the solution for the disk surface density with the total angular momentum J_d is given by

$$\Sigma = C\,\frac{J_d}{r^4\Omega}(x^A - x)^{1/\delta}, \quad x = \frac{B\,y}{(1-A)C^\delta}, \quad C = \left[\frac{2\pi}{a}\int_0^1 (x^A - x)^{1/\delta}\,\frac{dx}{x}\right]^{-1}. \tag{7.59}$$

7.3.4 Disk heating by viscous dissipation

The dissipation energy due to viscosity per unit volume per unit time, ϵ, is given for axi-symmetric Keplerian disks by

$$\epsilon = \Pi'_{ik}\frac{\partial v_i}{\partial x_k} = \rho\nu\left(r\frac{d\Omega}{dr}\right)^2 = \frac{9}{4}\rho\nu\Omega^2. \tag{7.60}$$

Vertical integration gives the heating rate per unit area of the disk. It is balanced by the radiative cooling rate at the upper and lower disk surfaces given by $2\sigma T_s^4$, where T_s is the temperature at the disk surface. Furthermore, assuming a steady accretion disk, the surface temperature is obtained as

$$T_s = \left(\frac{3GM_*\dot{\mathcal{M}}}{8\pi\sigma r^3}\right)^{1/4} \propto r^{-3/4}. \tag{7.61}$$

It is assumed above that $r \gg r_{\rm in}$. The mass accretion rate of $\dot{\mathcal{M}} = 10^{-8}\mathrm{M}_\odot/\mathrm{yr}$ gives $T_s =$ 90K at 1au. This is lower than the temperature of the Hayashi model for protoplanetary disks, which is heated by the stellar radiation[38]. Since T_s has a steeper radial gradient

[38] The ratio between two temperatures given by the equations (7.61) and (7.28) is determined by the ratio of the viscous heating rate to that by the stellar radiation, which given by

$$\frac{6GM_*\dot{\mathcal{M}}/R}{L_*} \simeq 10^{-2}\left(\frac{\dot{\mathcal{M}}}{10^{-8}\mathrm{M}_\odot/\mathrm{yr}}\right)\left(\frac{M_*}{M_\odot}\right)\left(\frac{L_*}{L_\odot}\right)^{-1}\left(\frac{r}{1\mathrm{au}}\right)^{-1}.$$

than that of the Hayashi model, the viscous heating can dominate the stellar radiation heating at an inner radius with $R \ll 1$au.

Optically thick protoplanetary disks can have an inner temperature much higher than T_s. When the energy dissipation due to viscosity is concentrated to the disk midplane or uniformly distributed (i.e., the dissipation rate $\propto \rho$), the midplane temperature is approximately given by $\sim \tau^{1/4} T_s$, where the vertical optical depth τ is defined by $\kappa_R \Sigma$ and κ_R is the Rosseland mean opacity of the disk material. However, if the viscous heating occurs only at the disk surface, the midplane temperature is similar to T_s (Mori et al. 2019).

We also describe the energy balance for unit mass of viscous accretion disks. Taking the scalar product of the equation (7.30) with $\boldsymbol{v}$, we obtain the equation for the kinetic energy per unit mass as

$$\begin{aligned}\left[\frac{\partial}{\partial t} + (\boldsymbol{v} \cdot \mathbf{grad})\right]\left(\frac{v^2}{2}\right) = \; & -\frac{1}{\rho}\boldsymbol{v} \cdot \mathbf{grad}\, p - \boldsymbol{v} \cdot \mathbf{grad}\left(-\frac{GM_*}{r}\right) \\ & + \frac{1}{\rho}\mathrm{div}\,(\boldsymbol{v} \cdot \boldsymbol{\Pi'}) - \frac{1}{\rho}\frac{\partial v_i}{\partial x_k}\Pi'_{ik}. \qquad (7.62)\end{aligned}$$

In the above equation, the term of the pressure gradient can be neglected since $c_s^2 \sim p/\rho$ is much smaller than the rotational energy of $v_\phi^2/2 = r^2\Omega^2/2$ for standard thin accretion disks. Furthermore, since $|v_R| \ll |v_\phi|$, the kinetic energy of $v^2/2$ is replaced by $v_\phi^2/2$ and only $\Pi'_{r\phi}$ should be considered for the viscous stress tensor. Thus the equation (7.62) can be rewritten for two-dimensional disk as

$$v_r \frac{\partial}{\partial r}\left(\frac{r^2\Omega^2}{2}\right) = -v_r \frac{\partial}{\partial r}\left(-\frac{GM_*}{r}\right) + \frac{1}{\Sigma r}\frac{\partial}{\partial r}\left(-\frac{3}{2}\Sigma\nu r^2\Omega^2\right) - \frac{9}{4}\nu\Omega^2. \qquad (7.63)$$

The meaning of each term is obvious. The left-hand side is the energy increase due to the acceleration of the disk rotation for accretion disks with $v_r < 0$. The first term on the right is the work done by the stellar gravity. We find that the ratio between these terms is $1/2:1$, as also derived from the virial theorem. Half of the stellar gravitational energy remains. The second term on the right represents the net energy gain due to the work done by the torques exerted by the inner and outer parts, and the third term is the loss of the kinetic energy due to viscous dissipation. For steady accretion disks with $\dot{\mathcal{M}} \equiv -2\pi r\Sigma v_r = 3\pi\Sigma\nu$, $v_r = -\frac{3\nu}{2r}$ and the third term is equal to $v_r(\partial r^2\Omega^2/\partial r)$. Then, we find that these four terms are in the ratio $1/2:1:1:$-$3/2$. The viscous heating costs the three times the remaining stellar gravitational energy. The deficit is supplied by the net work by the inner and outer viscous torques. This energy balance for steady accretion disks is governed by the Keplerian rotation law and is independent of the property of viscosity. It is also applicable to particle disks such as planetesimal disks or planetary rings.

8 Fundamentals of Relativistic Fluid Dynamics

8.1 Minimal elements in the special theory of relativity

There are flows with velocities close to the speed of light in space.

An example is a relativistic jet ejected from a compact object. To study such flows, fluid mechanics based on the theory of relativity is necessary. In this chapter, we will derive basic equations of relativistic fluid dynamics and see how the Newtonian fluid dynamics is extended. Before we derive the hydrodynamic equations, let us briefly review the basics of special relativity[39].

(a) Principle of relativity and Lorentz transformation

Einstein's principle of relativity consists of the following two principles.

- **Principle of relativity**: The laws of physics are identical in all inertial frames of reference. Therefore, the speed of light propagating in a vacuum must be the same from the point of view of any inertial frames.
- **The upper limit on velocity**: In the theory of relativity, an interaction between two objects occurs when information propagates from one object to the other at a finite speed. There is an upper limit to this propagation speed and the speed of objects, and the upper limit is the speed of light[40].

In Newtonian mechanics, time is absolute regardless of the frame of reference. Therefore, the velocity of objects and the speed of light vary with each inertial frame and are relative quantities. On the other hand, in the theory of relativity, absolute time is not allowed because the speed of light is assumed to be constant, and different times are used in each inertial frame. The time and space coordinates of the two inertial frames are transformed into each other by the Lorentz transformation. If, for an inertial coordinate system (t, x, y, z), there is another inertial system (t', x', y', z') with parallel spatial axes moving in the x direction at V, the Lorentz transformation, which is a coordinate transformation between these inertial systems, is given by

$$x = \frac{x' + Vt'}{\sqrt{1 - V^2/c^2}}, \qquad y = y', \qquad z = z', \qquad t = \frac{t' + Vx'/c^2}{\sqrt{1 - V^2/c^2}}. \tag{8.1}$$

From this Lorentz transformation, we can see that the length of a rod moving at the velocity V contracts by $\sqrt{1 - V^2/c^2}$ compared to when it is at rest (Lorentz contraction) and that a clock in motion advances more slowly than a clock at rest.

[39] The explanation here is minimal. For more details, please refer to textbooks on special relativity

[40] This is shown by the above principle of relativity. If it is possible for an object to move faster than the speed of light, it is also possible for it to move at the same speed as light, in which case light will not propagate from the object's point of view. This contradicts the principle of relativity, which requires the speed of light to be constant.

(b) World interval

In a four-dimensional space including the time axis, events are represented by points (world points). indexworld point In addition, the movement of particles and the propagation of light are represented by curves (world lines) in four-dimensional space. A particle at rest in an inertial frame moves along a straight line parallel to the time axis in four-dimensional space. The distance, s_{12}, between two events (t_1, x_1, y_1, z_1) and (t_2, x_2, y_2, z_2) in a four-dimensional space is called the world interval and defined by

$$s_{12} = \sqrt{c^2(t_2 - t_1)^2 - (x_2 - x_1)^2 - (y_2 - y_1)^2 - (z_2 - z_1)^2}. \tag{8.2}$$

The world interval ds between two adjacent events is given by[41].

$$ds^2 = c^2dt^2 - dx^2 - dy^2 - dz^2. \tag{8.3}$$

The world interval ds between two points on the world line of light gives ds = 0 by definition. If the world interval between two events vanishes in a certain inertial frame, it also vanishes in any other inertial frame from the principle of the invariance of the light speed. According to the principle of relativity, all inertial frames are equivalent, so the world interval ds between two points must have the same value in each inertial frame. The Lorentz transformation satisfies this requirement.

(c) Four-vectors and covariant form

It is convenient to write relativistic equations in terms of 4-vectors (or 4-tensors). An equation expressed with 4-tensors is invariant to Lorentz transformations and said to be covariant,

- Examples of 4-vectors
  - The four-dimensional coordinate vector $x^i = (x^0, x^1, x^2, x^3) = (ct, x, y, z)$. Generally, the component with index 0 is called the time component, and the others are called the space components.
  - The four-dimensional infinitesimal coordinate interval vector $dx^i = (cdt, dx, dy, dz)$.
  - The charge current density 4-vector $j_e^i = (c\rho_e, \boldsymbol{j}_e)$. Its time component is the flux density in the direction of the time axis.
  - The electromagnetic 4-potential $A^i = (\phi/c, \boldsymbol{A})$ (SI units).
- World interval and metric tensor

  The world interval ds is invariant in the Lorentz transformation and is a scalar. It is written in the covariant form

$$ds^2 = g_{ik}dx^i dx^k, \tag{8.4}$$

[41] In another style, the world interval ds^2 is defined with the opposite sign as $ds^2 = -c^2dt^2 + dx^2 + dy^2 + dz^2$.

where g_{ik} is the metric tensor. The indices i and k are summed from 0 to 3, respectively. In a Cartesian coordinate system, the components of the metric tensor g_{ik} is zero except for the diagonal ones, which are given by

$$g_{00} = 1, \qquad g_{\alpha\alpha} = -1 \qquad (\alpha = 1, 2, 3). \tag{8.5}$$

- Contravariant and covariant components

  There are two types of 4-vectors, covariant and contravariant components. The covariant component is expressed as A_i, and the contravariant component is expressed as A^i, distinguishing between the upper and lower indices. The above examples of the 4-vectors are contravariant components. An example of a covariant component is the four-dimensional gradient $d\phi/dx^i$ of a scalar ϕ. (A scalar is an invariant quantity in Lorentz transformations.) The contravariant and covariant components are related to each other via the metric tensor as

$$A_i = g_{ik}A^k, \qquad A^i = g^{ik}A_k, \tag{8.6}$$

  where g_{ik} is the covariant component of the metric tensor and g^{ik} is the contravariant one. Generally, these two components of the metric tensor are inverse matrices of each other, and they are equal in Cartesian coordinate systems. As shown in the above equation, the metric tensor moves the indices up and down. As in Eq. (8.4), for the index k repeated twice, the sum from 0 to 3 is taken, and this sum is called a reduction. Reductions are always taken for a pair of the indices of the covariant and contravariant components.

- Lorentz transformation of 4-vectors

  The Lorentz transformation of contravariant components of a 4-vector is the same as that of the four-dimensional coordinates of Eq. (8.1) and is given by

$$A^0 = \gamma(A'^0 + VA'^1/c), \quad A^1 = \gamma(A'^1 + VA'^0/c), \quad A^2 = A'^2, \quad A^3 = A'^3, \tag{8.7}$$

  where $\gamma = 1/\sqrt{1 - V^2/c^2}$ is the Lorentz factor. On the other hand, a covariant component is transformed as[42].

$$A_0 = \gamma(A'_0 - VA'_1/c), \quad A_1 = \gamma(A'_1 - VA'_0/c), \quad A_2 = A'_2, \quad A_3 = A'_3 \tag{8.8}$$

  The reduced product of a covariant vector and a contravariant vector (scalar product) A_iB^i is invariant to Lorentz transformations, and is a scalar.

(d) Four-velocity

The contravariant component u^i of the four-dimensional velocity vector (4-velocity)

[42] Generally, between two inertial frames moving in arbitrary directions relative to each other, the transformations are written as $A^i = (\partial x^i/\partial x'^k)A'^k$ and $A_i = (\partial x'^k/\partial x^i)A'_k$, These expressions of the vector transformation can also be used for arbitrary coordinate transformations in the general theory of relativity.

is defined by[43]

$$u^i = \frac{dx^i}{ds}. \tag{8.9}$$

By the definition of the world interval ds, the square of the 4-velocity gives $u_i u^i = 1$. Thus, the 4-velocity is a unit vector tangent to the world line. The Lorentz factor between the laboratory system and the local system moving with the fluid is given by $\gamma = 1/\sqrt{1 - v^2/c^2}$, where $v = \sqrt{v_\alpha^2}$. Using this Lorenz factor and Eq. (8.3), we have $ds = cdt/\gamma$, and the relation between the 4-velocity and the three-dimensional velocity v_α is written as [44]

$$u^0 = \gamma, \qquad u^\alpha = v_\alpha \gamma / c \qquad (\alpha = 1, 2, 3). \tag{8.10}$$

The 4-momentum of a particle with the rest mass m_0 and the velocity v_α is given by $m_0 c u^i$. The time component of the 4-momentum is equal to $1/c$ times the energy of the particle (including its rest mass energy), and the space component is its momentum. The Lorentz factor γ included in u_i indicates an increase in the inertial mass.

8.2 Equation of continuity

In the theory of relativity, the mass of a particle depends on its velocity and is not invariant. Therefore, we write down the conservation of the number of particles that make up the fluid as an equation of continuity[45]. In Newtonian mechanics, the 3D particle number flux density is given by $\boldsymbol{j} = n\boldsymbol{v}$, where n is the particle number density. As a natural extension, the 4-vector of the particle number flux density j^i in the theory of relativity is defined by

$$j^i = n u^i. \tag{8.11}$$

In the theory of relativity, the number density n (the particle number per unit volume) is defined using the unit volume in a local inertial frame where the fluid is at rest. Consider a fluid particle having a unit volume in the fluid rest frame. In the laboratory system where the fluid particle moves with the velocity $\boldsymbol{v}$, this fluid particle has a volume of $1/\gamma = \sqrt{1 - v^2/c^2}$ due to the Lorentz contraction. Thus, the number density in the laboratory system is larger by γ than that in the fluid rest frame. Although the number density varies for each inertial frame, the number density n is a scalar due to the definition of the unit volume in the fluid rest frame.

The time component of the 4-vector of the number flux density is $j^0 = n\gamma$ from Eq (8.10), which equals the number density in the laboratory system[46]. Similarly, the

[43] In another style, the 4-velocity is defined by $dx^i/d\tau$, where the proper time τ along the world line is defined by $d\tau = ds/c$.

[44] In special relativity, all three-dimensional vectors have a lower index, and the indices are written in Greek letters in our style.

[45] At high temperatures, particles and antiparticles are created by pair creation, and pair annihilation also occurs, so the total number of particles is not invariant. Therefore, the more general equation of continuity uses the difference in the number between particles and antiparticles, which does not change due to the pair creation or pair annihilation.

[46] It can also be considered to be $1/c$ times the number flux density in the direction of the x_0 axis.

space component is $j^\alpha = n\gamma v_\alpha/c$, which equals $1/c$ times the 3D particle number flux density in the laboratory system.

In relativistic fluid dynamics, the equation of continuity for the number density is written down as[47]

$$\frac{\partial(nu^i)}{\partial x^i} = 0. \tag{8.12}$$

This is expressed in the three-dimensional form

$$\frac{\partial(n\gamma)}{\partial t} + \frac{\partial(n\gamma v_\alpha)}{\partial x_\alpha} = 0. \tag{8.13}$$

Since the number density in the laboratory system is $n\gamma$, the obtained equation of continuity (8.13) is the same as that in Newtonian mechanics.

8.3 Energy-momentum tensor of ideal fluid

As we saw in Chapter 1, the equation of motion and the energy equation of fluid mechanics correspond to the conservation of momentum and energy, respectively. The equation of energy conservation (or momentum conservation) is expressed in terms of the energy density (or momentum density) and its flux density. Therefore, once we obtain these expressions in the theory of relativity, we can readily write down the conservation equation. In this section, we derive such relativistic expressions for ideal fluids.

In the theory of relativity, the energy density, momentum density, and their flux densities are the components of the four-dimensional energy-momentum tensor T^{ik}. The component T^{00} is the energy density. A vector, $T^{\alpha 0}$ (α =1,2,3), corresponds to c times the 3D momentum density[48]. A vector of $T^{0\alpha}$ is $1/c$ times the energy flux density, and the tensor of $T^{\alpha\beta}$ is the momentum flux density[49]. The energy-momentum tensor T_{ik} is symmetric.

Let us first find the energy-momentum tensor of an ideal fluid for the inertial frame of reference in which the fluid is at rest. For a fluid at rest, the momentum density and the energy flux density vanish (see Eq. [1.37]). The momentum flux density tensor is diagonal, and each diagonal component is given by the pressure p (see Eq. [1.19]). Therefore, the energy-momentum tensor of an ideal fluid at rest is

$$T^{ik} = \begin{pmatrix} e & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix}. \tag{8.14}$$

Note that the internal energy per unit volume e also includes the rest mass energy.

[47] The vanishing of the 4-divergence of the number flux density j_i indicates that, for any 4-dimensional volume, the number of the particle's world lines that enter the volume equals the number of the lines that exit outward.

[48] These energy and momentum densities are defined in the laboratory system.

[49] Since the components T^{i0} can be regarded as the flux densities of the energy and momentum in the direction of the x_0 axis, we can say that T^{ik} is the 4-tensor of the energy-momentum flux densities.

The general expression for the energy-momentum tensor of an ideal fluid in arbitrary inertial frames is also easy to find. The general expression of T_{ik} has a quadratic term of the 4-velocity u_i. And, it must be equal to Eq. (8.14) for the fluid rest frame, where $u^0 = 1$ and $u^\alpha = 0$. Therefore, the general expression for the energy-momentum tensor of an ideal fluid is given by

$$T^{ik} = hu^i u^k - pg^{ik}, \tag{8.15}$$

where $h = e + p$ is the enthalpy per unit volume[50]. Using Eq. (8.10), each component of T_{ik} is expressed with the 3D velocity v_α as[51].

$$T^{00} = h\gamma^2 - p, \qquad T^{\alpha 0} = T^{0\alpha} = h\gamma^2 v_\alpha / c, \qquad T^{\alpha\beta} = h\gamma^2 v_\alpha v_\beta / c^2 + p\delta_{\alpha\beta}. \tag{8.16}$$

We also check that T_{ij} is consistent with Newtonian fluid mechanics in the non-relativistic limit ($v \ll c$). The density ρ_{nr} in Newtonian mechanics is equal to $m_0 n\gamma$, where m_0 is the rest mass of the particles, and $n\gamma$ is the number density in the laboratory frame. The internal energy e_{nr} in non-relativistic Newtonian mechanics is related to the relativistic internal energy e as $e\gamma = \rho_{nr}c^2 + e_{nr}$. Since the Lorentz factor in the non-relativistic limit is given by $\gamma = 1 + \frac{1}{2}v^2/c^2$, we have $h\gamma^2 = \rho_{nr}c^2 + \frac{1}{2}\rho_{nr}v^2 + e_{nr} + p$. Thus, leaving the terms of up to $1/c$ in Eq. (8.16), we obtain the approximate expression of T_{ik} in the non-relativistic limit as

$$\begin{aligned} &T^{00} = \rho_{nr}c^2 + \frac{1}{2}\rho_{nr}v^2 + e_{nr}, \qquad T^{\alpha 0} = T^{0\alpha} = (\rho_{nr}c^2 + \frac{1}{2}\rho_{nr}v^2 + e_{nr} + p)v_\alpha / c, \\ &T^{\alpha\beta} = \rho_{nr} v_\alpha v_\beta + p\delta_{\alpha\beta}. \end{aligned} \tag{8.17}$$

The tensor $T^{\alpha\beta}$ is equal to the Newtonian momentum flux density of Eq. (1.19). Since T^{00} includes the rest-mass energy, we can see that T^{00} is also consistent with the Newtonian energy density.

The vector $T^{\alpha 0}$ ($= T^{0\alpha}$) is equal to c times the Newtonian momentum density with accuracy up to the c^1 term. $T^{\alpha 0}$ is also the energy flux density. Noting that $T^{\alpha 0}$ includes the advection term due to the rest mass energy, we see that the other terms equals $1/c$ times the Newtonian energy flux density (see Eq. [1.37]).

8.4 Hydrodynamic equations in special relativity

In special relativity, the conservation equations for the energy and momentum are expressed with 4-divergence as

$$\frac{\partial T_i^k}{\partial x^k} = \frac{\partial (hu_i u^k)}{\partial x^k} - \frac{\partial p}{\partial x^i} = 0, \tag{8.18}$$

[50]We recall again that the theory of relativity defines all thermodynamic quantities (n, e, h, p, and volume) in the fluid rest frame.

[51]Equation (8.16) can be derived by the Lorentz transformation of T^{ik} in the fluid rest frame of Eq. (8.14). The transformation formula for a tensor can be derived from that of the tensor product of vectors.

where T_i^k is the mixed component of the energy-momentum tensor given by $T_i^k = g_{il}T^{lk}$. The time component of Eq. (8.18) is the equation of energy conservation, and the space components are the equations of momentum conservation. Using Eq. (8.16), we obtain the three-dimensional form of the equation of energy conservation as

$$\frac{\partial(h\gamma^2 - p)}{\partial t} + \frac{\partial(h\gamma^2 v_\alpha)}{\partial x_\alpha} = 0, \tag{8.19}$$

and the 3D form of the equation of momentum conservation is written as

$$\frac{\partial(h\gamma^2 v_\alpha)}{\partial t} + \frac{\partial(h\gamma^2 v_\alpha v_\beta + c^2 p\delta_{\alpha\beta})}{\partial x_\beta} = 0. \tag{8.20}$$

These conservation equations and the equation of continuity (8.13) are basic equations of special relativistic hydrodynamics. The number of the equations is five, which agree with the number of the independent variables (i.e., three components of the velocity and two independent thermodynamic quantities).

We also describe the equations derived from these basic equations of Eqs. (8.18) and (8.13). Taking the scalar product of u_i and Eq. (8.18) and noting that $u_i u^i = 1$ and $u_i \partial u^i/\partial x^k = 0$, we have

$$\frac{\partial(hu^k)}{\partial x^k} - u^k \frac{\partial p}{\partial x^k} = 0. \tag{8.21}$$

Using $hu^k = nu^k(h/n)$ and Eq. (8.12), we can rewrite Eq. (8.21) as

$$nu^k \left[\frac{\partial(h/n)}{\partial x^k} - \frac{1}{n}\frac{\partial p}{\partial x^k}\right] = 0. \tag{8.22}$$

Furthermore, using the thermodynamic relation for the entropy σ per unit volume, $Td(\sigma/n) = d(h/n) - (1/n)dp$, we have

$$u^k \frac{\partial(\sigma/n)}{\partial x^k} = \frac{\sigma/n)}{ds} = 0. \tag{8.23}$$

This is the adiabatic condition in special relativity, showing that the entropy per particle is invariant along the world line of the flow[52].

Next, consider the component perpendicular to u_i of Eq. (8.18). It is given by

$$\frac{\partial T_i^k}{\partial x^k} - u_i u^l \frac{\partial T_l^k}{\partial x^k} = 0. \tag{8.24}$$

Transforming this using Eqs. (8.18) and (8.21), we obtain Euler's equation in special relativity as

$$hu^k \frac{\partial u_i}{\partial x^k} = \frac{\partial p}{\partial x^i} - u_i u^k \frac{\partial p}{\partial x^k}. \tag{8.25}$$

We also derive the equation for steady flows. In this case, since the time derivative vanishes in Eqs. (8.13) and (8.19), we find that the particle number flux and energy flux across the cross-section of the flow tube are constant, respectively. That is,

$$n\gamma v_\alpha \, dS_\alpha = \text{constant}, \qquad h\gamma^2 v_\alpha \, dS_\alpha = \text{constant}, \tag{8.26}$$

[52]From this adiabatic condition and the equation of continuity (8.12), we also obtain the equation of entropy conservation as $\partial(\sigma u^k)/\partial x^k = 0$.

where dS_α is the cross-section vector of the flow tube. Since the ratio of these fluxes is also invariant, we have

$$h\gamma/n = \text{constant}. \tag{8.27}$$

This is Bernoulli's equation in special relativistic hydrodynamics.

8.5 Hydrodynamic equations in general relativity

8.5.1 Covariant form of the hydrodynamic equations

When we study fluid motion in a gravitational field strong enough to alter a flow with a velocity comparable to the speed of light, the general theory of relativity is necessary. General relativity is constructed based on the equivalence principle, which states that gravity and inertial force are equivalent. Thus, the gravitational field is due to an accelerated motion as well as the inertial force and is described by the metric tensor g_{ik} of spacetime. The metric tensor of the spacetime distorted by "gravitational sources" is governed by the Einstein equation. Generally, the metric tensor g_{ik} has non-zero, non-diagonal components, and its components depend on the four-dimensional coordinates. In this section, we consider the case where the metric tensor is a given one and derive hydrodynamic equations in the general theory of relativity. When the metric tensor g_{ik} is given, the hydrodynamic equations are easy to find. In fact, by using a given g_{ik}, the covariant equations expressed with 4-tensors derived in the previous section can be used with only a minor change. The changes concern the derivative of vectors and tensors.

In the curvilinear coordinate system used in general relativity, the covariant derivative is used to differentiate vectors and tensors. The covariant derivative of a vector A_i (or A^i) for a coordinate x_k is denoted by $A_{i;k}$ (or $A^i_{;k}$), and defined by

$$A_{i;k} = \frac{\partial A_i}{\partial x^k} - \Gamma^l_{ik} A_l, \qquad A^i_{;k} = \frac{\partial A^i}{\partial x^k} + \Gamma^i_{lk} A^l, \tag{8.28}$$

where Γ^l_{ik} is the Christoffel symbol, which is given by[53].

$$\Gamma^l_{ik} = \frac{1}{2} g^{lm} \left(\frac{\partial g_{mi}}{\partial x^k} + \frac{\partial g_{mk}}{\partial x^i} - \frac{\partial g_{ik}}{\partial x^m} \right). \tag{8.29}$$

Generally, in the covariant derivative of a tensor, terms of the Christoffel symbol are added for each index. For example, the covariant derivative of the mixed component T^i_j is given by

$$T^i_{j;k} = \frac{\partial T^i_j}{\partial x^k} - \Gamma^l_{jk} T^i_l + \Gamma^i_{lk} T^l_j. \tag{8.30}$$

The covariant derivative of a scalar does not include the Christoffel symbol term. Thus, it is the same as the conventional derivative. Using the covariant derivative instead of the conventional derivative, we can extend the equations to the covariant equations applicable

[53] Geometrically, the Christoffel symbol gives the changes in each component of vectors when they are made a parallel transport. A differential is the difference between two different points. To make it the difference at the same point, one of them must be made a parallel transport to the other point.

to general relativity. That is, the equation of continuity (8.12), the conservation equations of energy and momentum (8.18), and Euler's equation (8.25) are extended as

$$(nu^i)_{;i} = 0, \qquad T^k_{i;k} = 0, \qquad hu^k u_{i;k} = \frac{\partial p}{\partial x^i} - u_i u^k \frac{\partial p}{\partial x^k}. \tag{8.31}$$

In general relativity, the Christoffel symbol term included in the covariant derivative represents the effect of gravity. There is no change in the adiabatic condition (8.23) since it has the derivative of the scalar σ/n[54].

8.5.2 Hydrostatic equilibrium of a spherically symmetric object

The relativistic hydrostatic equation for a spherically symmetric self-gravitational object is obtained from Euler's equation (8.31). In the frame of reference where the object is at rest, we have $u^\alpha = 0$, $u^0 = 1/\sqrt{g_{00}}$. Furthermore, if we consider only the self-gravity of the static object, we obtain $g_{0\alpha} = 0$, $u_\alpha = 0$, and $u^0 u_0 = 1$.

Such a static object can be considered to be spherically symmetric[55]. Then, the r component of the Euler equation (8.31) is given by

$$-hu^0 \Gamma^0_{r0} u_0 = \frac{\partial p}{\partial r}. \tag{8.32}$$

Furthermore, from Eq. (8.29), we obtain $\Gamma^0_{r0} = \frac{1}{2} g^{00} dg_{00}/dr$ for a static and spherically symmetric gravitational field, and thus we have

$$\frac{1}{h}\frac{dp}{dr} = -\frac{1}{2}\frac{d\ln g_{00}}{dr}. \tag{8.33}$$

(ASIDE) Derivation of the TOV equation

By determining dg_{00}/dr from the Einstein equation, we can derive the Tolman–Oppenheimer–Volkoff equation (TOV equation). For a spherically symmetric, static gravitational field, the world interval ds is given by

$$ds^2 = g_{00}(r)\, c^2 dt^2 + g_{rr}(r)\, dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{8.34}$$

The 00-component of the Einstein equation given by $R^i_k - \frac{1}{2} R^l_l \delta^i_k = 8\pi G T^i_k/c^4$ is written down as (see §8.8 for the derivation)

$$\frac{1}{r^2}\frac{d}{dr}\left(\frac{r}{g_{rr}}\right) + \frac{1}{r^2} = \frac{8\pi G\, e}{c^4}. \tag{8.35}$$

This is the differential equation for g_{rr}. Solving it, we have

$$g_{rr} = -\left(1 - \frac{2GM(r)}{c^2 r}\right)^{-1}, \tag{8.36}$$

[54] In Newtonian mechanics, the adiabatic condition is not affected by the presence or absence of gravity.

[55] If the distributions of temperature and composition are spherically symmetric, the distributions of density and pressure should also be spherically symmetric.

where $M(r)$ is the mass inside r defined by Eq. (3.5), and the relativistic relation $\rho = e/c^2$ is used. The rr-component of the Einstein equation is given by (see §8.8)

$$\frac{1}{r^2 g_{rr}}\left(r\frac{d\ln g_{00}}{dr}+1\right)+\frac{1}{r^2}=-\frac{8\pi G\,p}{c^4}. \tag{8.37}$$

Solving this for $d\ln g_{00}/dr$ and using Eqs. (8.33) and (8.36), we finally obtain the TOV equation (4.37).

8.5.3 Hydrodynamic equations in a static and spherically symmetric gravitational field

Equation (8.31) is the covariant form of hydrodynamic equations, which are expressed with 4-vectors. To write down these equations with three-dimensional vectors, we extend Eq. (8.10) to the general relativistic equation. For simplicity, we will consider fluid motion in a static and spherically symmetric gravitational field ($\partial g_{ik}/\partial x^0 = 0$, $g_{0\alpha} = 0$) below.

In a static gravitational field, it is natural to define the three-dimensional velocity using the characteristic time at each position, $d\tau = \sqrt{g_{00}}dx^0/c$, as[56]

$$v^\alpha = \frac{dx^\alpha}{d\tau} = \frac{c}{\sqrt{g_{00}}}\frac{dx^\alpha}{dx^0}. \tag{8.38}$$

This is the contravariant component. The covariant component of the three-dimensional velocity is defined by $v_\alpha = -g_{\alpha\beta}v^\beta$[57]. Using the Lorentz factor defined by $\gamma = \sqrt{1 - v_\alpha v^\alpha/c^2}$, Eq. (8.4) is rewritten as $ds = cd\tau/\gamma$. Therefore, the relation between the 4-velocity and the 3D velocity of Eq. (8.38) is obtained as

$$u^0 = \gamma/\sqrt{g_{00}}, \qquad u^\alpha = v^\alpha\gamma/c. \tag{8.39}$$

Using this relation, each component of the energy-momentum tensor T^{ik} (8.15) is expressed in the three-dimensional velocity

$$T^{00} = \frac{h\gamma^2 - p}{g_{00}}, \qquad T^{\alpha 0} = T^{0\alpha} = \frac{h\gamma^2 v^\alpha}{c\sqrt{g_{00}}}, \qquad T^{\alpha\beta} = h\gamma^2 v^\alpha v^\beta/c^2 - pg^{\alpha\beta}, \tag{8.40}$$

where $g^{00}g_{00} = 1$ is used, which is valid in the static and spherically symmetric gravitational field. The mixed components are given by

$$T_0^0 = h\gamma^2 - p, \qquad T_0^\alpha = \sqrt{g_{00}}\,h\gamma^2 v^\alpha/c, \qquad T_\alpha^\beta = -h\gamma^2 v_\alpha v^\beta/c^2 - p\delta_\alpha^\beta. \tag{8.41}$$

Generally, the 4-divergence of a vector or tensor is obtained from Eqs. (8.28)-(8.30) as

$$A^i_{;i} = \frac{1}{\sqrt{-g}}\frac{\partial(\sqrt{-g}A^i)}{\partial x^i}, \qquad A^k_{i;k} = \frac{1}{\sqrt{-g}}\frac{\partial(\sqrt{-g}A^k_i)}{\partial x^k} - \frac{1}{2}\frac{\partial g_{kl}}{\partial x^i}A^{kl}, \tag{8.42}$$

[56]Note that this characteristic time is different from the characteristic time of the fluid-rest frame.

[57]In special relativity without the gravitational field, we can always use the Cartesian linear coordinate system with $g_{\alpha\alpha} = -1$. Then, we have $v_\alpha = v^\alpha$, that is, the covariant and contravariant components are equal to each other

where g is the determinant of the metric tensor g_{ik}. By rewriting the equation of continuity and the equation of motion in Eq. (8.31) with Eqs. (8.39)-(8.42), we can obtain the three-dimensional form of the hydrodynamic equations in a static and spherically symmetric gravitational field.

Let us also derive the general relativistic Bernoulli's equation for a static and spherically symmetric gravitational field. Similarly to the previous section, from the three-dimensional forms of the equation of continuity and the equation of energy conservation, we obtain the equations of constant particle number flux and energy flux across each cross section of a flow tube. From the ratio of these fluxes, we obtain

$$\sqrt{g_{00}}\, h\gamma/n = \text{constant}. \tag{8.43}$$

In the limit of a weak gravitational field, we obtain the relation between g_{00} and the Newtonian gravitational potential ϕ_g as[58]

$$g_{00} = 1 + 2\phi_g/c^2 \tag{8.44}$$

Substituting Eq. (8.44) into (8.43) and taking the non-relativistic limit of it, we obtain the Newtonian Bernoulli's equation (1.32) in a gravitational field.

Problem 45. Derive Eq. (8.42) from Eqs. (8.28)-(8.30). Also use the facts that the determinant g is negative and that its differential is given by $dg = gg^{ik}dg_{ik}$.

Problem 46. Derive general relativistic Bernoulli's equation (8.43) for a static and spherically symmetric gravitational field according to the above explanation. Also check that Eq. (8.43) is reduced to Newtonian Bernoulli's equation (1.32) in the non-relativistic limit.

8.6 Relativistic sound waves and shock waves

First, we consider sound waves in a relativistic case. Similar to §2.1 , we consider small perturbations in a uniform fluid in the fluid-rest frame. Here the '2elativistic" case means a case where the pressure is extremely high and comparable to the internal energy including the rest-mass energy. On the other hand, we can set as $\gamma = 1$ since the velocity is a small perturbation. Leaving only the first-order terms of perturbations, we can obtain the perturbation equations for those of the energy and momentum conservation (8.19) and (8.20) as

$$\frac{\partial e'}{\partial t} = -h\,\mathrm{div}\,\boldsymbol{v}', \qquad h\frac{\partial \boldsymbol{v}'}{\partial t} = -c^2\,\mathbf{grad}\,p', \tag{8.45}$$

where the quantities with dash $'$ represent the perturbations and the quantities without dash are the unperturbed ones. Eliminating the velocity perturbations $\boldsymbol{v}'$ from these

[58]In fact, substituting this relation into the general relativistic hydrostatic equation (8.33), we obtain the Newtonian hydrostatic equation in §3.1.

equations and using an adiabatic relation $p' = (\partial p/\partial e)_s\, e'$, we obtain the wave equation as $\partial^2 e'/\partial t^2 = c_s^2 \triangle e'$. The relativistic expression of the sound velocity is given by

$$c_s = c\sqrt{\left(\frac{\partial p}{\partial e}\right)_s}. \tag{8.46}$$

This is a natural extension from the Newtonian sound velocity since the mass density ρ is replaced by e/c^2 in the theory of relativity. In the ultra-relativistic limit (or in the high-energy limit), we have $c_s = c/\sqrt{3}$ since $p = e/3$ in this limit.

Next, we will discuss shock waves in the relativistic fluid dynamics. As in §2.3, we use a (local) frame of reference in which the shock front is at rest[59]. The x-axis is set perpendicular to the shock front, and the shock front is located at $x = 0$. Assume that v_x is positive and that there is no tangential velocity at the shock front. We label the pre-shock region with $x < 0$ as 1 and the post-shock region with a positive x as 2. The quantities in each region are represented by the subscripts 1 and 2.

As in the non-relativistic case, the particle number flux density $j^x = nu^x$, the momentum flux density T^{xx}, and the energy flux density T^{x0} are continuous at the discontinuous surface ($x = 0$). Using the three-dimensional expressions (8.10) and (8.16), these continuous conditions for the shock wave are given by[60]

$$\gamma_1 v_1/V_1 = \gamma_2 v_2/V_2 \equiv j, \tag{8.47}$$

$$h_1\gamma_1^2 v_1^2/c^2 + p_1 = h_2\gamma_2^2 v_2^2/c^2 + p_2, \tag{8.48}$$

$$h_1\gamma_1^2 v_1 = h_2\gamma_2^2 v_2, \tag{8.49}$$

where $V_i = 1/n_i$ the volume per particle in the fluid-rest frame and $\gamma_i = 1/\sqrt{1 - v_i^2/c^2}$ is the Lorentz factor in the region i. Eliminating $\gamma_i v_i$ from Eqs. (8.47) and (8.48), and solving it for j, we obtain[61]

$$j^2 = \frac{p_2 - p_1}{h_1 V_1^2 - h_2 V_2^2}\, c^2. \tag{8.50}$$

Furthermore, equations (8.47) and (8.49) yields

$$h_1^2\gamma_1^2 V_1^2 = h_2^2\gamma_2^2 V_2^2. \tag{8.51}$$

From $j = \gamma_i v_i/V_i$, we obtain another expression for the Lorentz factor as

$$\gamma_i^2 = 1 + j^2 V_i^2/c^2 \tag{8.52}$$

[59]Furthermore, by using a local inertial frame, gravity can be eliminated locally. Note that the metric tensor is continuous even at the shock front.

[60]We recall again that h and e used in this chapter are quantities per unit volume (of the fluid-rest frame) although they are defined as quantities per unit mass in Chapter 2. Another difference from Chapter 2 is that they include the rest-mass energy.

[61]Noting that $hV = h/n \simeq m_0c^2$ in the non-relativistic limit and that j and V are the particle number flux and the volume per particle, respectively, we find that Eq. (8.50) is equal to the Newtonian relation (2.33) in this limit. Similarly, Eq. (8.54) is found to equal Eq. (2.35) in the non-relativistic limit.

Eliminating the Lorentz factors in Eq. (8.51) using the expression (8.52), we obtain another expression for j as

$$j^2 = \frac{h_2^2 V_2^2 - h_1^2 V_1^2}{h_1^2 V_1^4 - h_2^2 V_2^4}\, c^2. \tag{8.53}$$

From the equivalence between this and Eq. (8.50), a relation between the thermodynamic quantities is obtained as

$$p_2 - p_1 = \frac{h_2^2 {V_2}^2 - h_1^2 {V_1}^2}{h_2 {V_2}^2 + h_1 {V_1}^2}, \qquad \text{or} \qquad h_1(e_1 + p_2)V_1^2 = h_2(e_2 + p_1)V_2^2. \tag{8.54}$$

This corresponds to the non-relativistic relation (2.35). Expressing the enthalpy with p and V in Eq. (8.54) using the equation of state, and solving it for p_2, we can plot the curve of the shock adiabat in the p-V plane. This relativistic shock adiabat is called **Taub's adiabat** and Eq. (8.54) is the equation of Taub's adiabat.

The velocities in each region can also be expressed with thermodynamic quantities on both sides. In fact, a simple transformation of Eqs. (8.48) and (8.49) yields[62]

$$\frac{v_1}{c} = \sqrt{\frac{(p_2 - p_1)(e_2 + p_1)}{(e_2 - e_1)(e_1 + p_2)}}, \qquad \frac{v_2}{c} = \sqrt{\frac{(p_2 - p_1)(e_1 + p_2)}{(e_2 - e_1)(e_2 + p_1)}}. \tag{8.55}$$

In the limit of a weak shock, both v_i approach the sound velocity of Eq. (8.46) since quantities on both sides are approximately equal. On the other hand, in the limit of a strong shock where $e_2 \gg e_1$, and when the ultrarelativistic equation of state, $p_2 = e_2/3$, holds in the post-shock region, v_1 approaches c and v_2 approaches $c/3$.

The relative velocity between both sides, v_{12} is obtained from the addition formula of velocities and Eq. (8.55) as

$$v_{12} = \frac{v_1 - v_2}{1 - v_1 v_2/c^2} = c\sqrt{\frac{(p_2 - p_1)(e_2 - e_1)}{(e_2 + p_1)(e_1 + p_2)}}. \tag{8.56}$$

In the ultra-relativistic strong shock limit described above, v_{12} also approaches c.

Problem 47. Derive Eqs. (8.52), (8.53), and (8.54).

Problem 48. Derive Eq. (8.55).

Problem 49. Show that, for the ultra-relativistic strong shock limit, quantities in the post-shock region are given by[63]

$$\gamma_2 = \sqrt{9/8}, \qquad \gamma_2 n_2 = 3\gamma_1 n_1, \qquad e_2 = 3p_2 = 3h_2/4 = 2\gamma_1^2 h_1. \tag{8.57}$$

[62] For the derivations, it is useful to set the velocity and Lorentz factor on both sides as $v_i/c = \tanh\phi_i$ and $\gamma_i = \cosh\phi_i$, respectively. Also, note the relation between the hyperbolic function $\cosh^2\phi_i - \sinh^2\phi_i = 1$. Similarly, the expression for γ_i can be easily obtained.

[63] In the non-relativistic strong shock limit, the number density ratio n_2/n_1 has a upper limit. On the other hand, in the ultra-relativistic strong shock limit, the ratio n_2/n_1 is proportional to γ_1 and increases infinitely.

Problem 50. Show that the Lorentz factor corresponding to the relative velocity between the both sides, $\gamma_{12} = 1/\sqrt{1 - v_{12}^2/c^2}$, satisfies $\gamma_{12} = \gamma_1\gamma_2(1 - v_1 v_2/c^2)$. (Firstly, show that $\phi_{12} \equiv \tanh^{-1}(v_{12}/c) = \phi_1 - \phi_2$.) Also, show that $\gamma_{12} = \gamma_1/\sqrt{2}$ for the ultra-relativistic strong shock limit.

8.7 Ultra-relativistic blast waves

In §5.3, we derived Sedov's solution, which is a non-relativistic solution to the blast wave. However, when the explosion has a quite high energy and the propagation speed of the blast wave is comparable to the speed of light, the relativistic fluid dynamics must be used. In this section, we will consider an ultra-relativistic blast wave where the propagation speed of the blast wave is extremely close to the speed of light and its Lorentz factor is large. The solution for an ultra-relativistic blast wave is known as the Blandford-McKee solution. The solution of ultrarelativistic blast waves helps us to understand a shock wave generated by the collision of a relativistic jet from black holes with the interstellar medium.

As in the non-relativistic Sedov's solution, we consider a spherical shock wave propagating in a homogeneous medium at rest and an adiabatic flow inside the shock. In an ultra-relativistic blast wave, the propagation speed of the shock wave is approximately the speed of light c[64]. Therefore, the radius R of the spherical shock is given by ct. The propagation velocity is equal to the relative velocity between the shock and the outer medium denoted by v_1 in the previous section, and the corresponding Lorentz factor is given by γ_1, which is much larger than unity.

We fist estimate the number density n_2 just inside the shock. Let h_1 be the enthalpy per unit volume of the outer medium, and let n_1 be the number density of the medium. Although the thermodynamic quantities at the post-shock are given by Eq. (8.57), it should be noted that those are defined in the frame of reference where the shock is at rest. The number density on the post-shock side, n_2' in the frame where the medium is at rest is given by

$$n_2' = \gamma_{12} n_2 = 2\gamma_1^2 n_1, \tag{8.58}$$

where γ_{12} is the Lorentz factor corresponding to the relative velocity v_{12}. Since this equation shows that $n_2 \gg n_1$, we find that for an ultra-relativistic blast wave, the gas is concentrated within an extremely thin spherical shell just inside of the shock.

Assuming the uniform number density distribution over the shell volume $4\pi R^2 \Delta R$, the shell thickness ΔR can be estimated as

$$\Delta R \simeq \frac{n_1}{3n_2'} R = \frac{R}{6\gamma_1^2}. \tag{8.59}$$

As the blast wave propagates through the medium, the Lorentz factor γ_1 gradually decreases. Let us examine time evolution of γ_1. The energy density T_{00} in the spherical

[64] In general, for relativistic blast waves, the hydrodynamic equations also includes the speed of light, so there are two independent dimensionless variables in this problem. Therefore, the dimensional analysis done for Sedov's solution is not applicable. However, for ultra-relativistic blast waves, the Lorentz factor γ_1 is the only one dimensionless variable and thus a self-similar solution exists.

shell is estimated as $h_2\gamma_{12}^2$ from Eq. (8.16). Multiplying this by the volume of the shell and using Eq. (8.57), the total energy of the blast wave is estimated to be $8\pi R^3 h_1 \gamma_1^2/9$. Since this is equal to the explosion energy E, we obtain time evolution of γ_1 as

$$\gamma_1^2 \simeq \frac{9E}{8\pi h_1 R^3} = \frac{9E}{8\pi h_1 c^3 t^3}. \tag{8.60}$$

As the Lorentz factor γ_1 of decreases, the pressure, number density, energy density of the shell all decrease, and the relative thickness $\Delta R/R$ of the shell increases. Exactly speaking, it is an overestimate to assume that the (average) energy density in the shell is equal to the post-shock value. In a more accurate estimate, γ_1^2 is about twice of Eq. (8.60) (Blandford & McKee 1976).

8.8 Appendix: Christoffel symbol and Ricci tensor in a static and spherically symmetric gravitational field

In this section, we derive the expressions for the Christoffel symbol and the Ricci tensor with the metric tensor g_{ik} for a static and spherically symmetric gravitational field. The metric tensor of a spherically symmetric coordinate system $(x^0, x^1, x^2, x^3) = (ct, r, \theta, \phi)$ in a spherically symmetric and static gravitational field is given by the diagonal matrix of Eq. (8.34). The diagonal components, g_{00} and g_{11}, are functions of only $r(= x_1)$ and the other components are

$$g_{22} = 1/g^{22} = -r^2, \qquad g_{33} = -r^2 \sin^2\theta. \tag{8.61}$$

The contravariant components are given by $g^{ii} = 1/g_{ii}$, where the summation over i is not taken. Using the metric tensor, each component of the Christoffel symbol is calculated from the definition of Eq. (8.29) as

$$\begin{aligned}
&\Gamma^1_{00} = -\frac{g'_{00}}{2g_{11}}, \qquad &&\Gamma^0_{10} = \Gamma^0_{01} = \frac{g'_{00}}{2g_{00}}, \qquad &&\Gamma^1_{11} = \frac{g'_{11}}{2g_{11}},\\
&\Gamma^1_{22} = \frac{r}{g_{11}}, &&\Gamma^2_{12} = \Gamma^2_{21} = \frac{1}{r}, &&\Gamma^3_{13} = \Gamma^3_{31} = \frac{1}{r},\\
&\Gamma^1_{33} = \frac{r\sin^2\theta}{g_{11}}, &&\Gamma^2_{33} = -\sin\theta\cos\theta, &&\Gamma^3_{23} = \Gamma^3_{32} = \cot\theta,
\end{aligned} \tag{8.62}$$

where the dash $'$ represents the derivative with respect to r. The other components of the Christoffel symbol are zero.

The Ricci tensor is given by $R_{ik} = \partial\Gamma^l_{ik}/\partial x^l - \partial\Gamma^l_{il}/\partial x^k + \Gamma^l_{ik}\Gamma^m_{lm} - \Gamma^m_{il}\Gamma^l_{km}$, which are diagonal in the spherical symmetric and static gravitational field. The diagonal components of R_{ik} is obtained as

$$\begin{aligned} R_{00} &= -\frac{g_{00}''}{2g_{11}} + \frac{g_{00}'g_{11}'}{4g_{11}^2} + \frac{{g_{00}'}^2}{4g_{00}g_{11}} - \frac{g_{00}'}{rg_{11}}, \\ R_{11} &= -\frac{g_{00}''}{2g_{00}} + \left(\frac{g_{00}'}{2g_{00}^2}\right)^2 + \frac{g_{11}'}{2g_{11}}\left(\frac{g_{00}'}{2g_{00}} + \frac{2}{r}\right), \\ R_{22} &= \left(\frac{r}{g_{11}}\right)' + 1 + \frac{r}{g_{11}}\left(\frac{g_{00}'}{2g_{00}} + \frac{g_{11}'}{2g_{11}}\right), \\ R_{33} &= R_{22}\sin^2\theta. \end{aligned} \tag{8.63}$$

The diagonal component R_i^i of the mixed component is given by $R_i^i = g^{ij}R_{ji}$, where the summation over i is not taken. The scalar curvature R is the diagonal sum of the mixed components of the Ricci tensor and obtained as

$$R = -\frac{g_{00}''}{g_{00}g_{11}} + \frac{g_{00}'g_{11}'}{2g_{00}g_{11}^2} + \frac{{g_{00}'}^2}{2g_{00}^2g_{11}} - \frac{2g_{00}'}{rg_{00}g_{11}} - \frac{2}{r^2}\left(\frac{r}{g_{11}}\right)' + \frac{2}{r^2}. \tag{8.64}$$

Using the mixed component R_k^i and the scalar curvature R, we can obtain the left-hand sides of Einstein's equations (8.35) and (8.37) for a spherically symmetric and static gravitational field.

Index