Fundamentals of Planet Formation Theory

Contents

1 Gravitational Contraction of Molecular Clouds and Jeans Instability

In view of the self-gravitational contraction of molecular clouds, we study the conditions for the self-gravitational fragmentation of uniform gas (the gravitational instability of uniform gas). This problem is called the Jeans instability.

1.1 Basic equations

We describe equations for self-gravitating fluids. Heat transfer, viscosity, rotation, and magnetic field are neglected.

- Equation of continuity

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{v}) = 0. \tag{1.1}$$

- Euler's equation.

$$\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v} \cdot \nabla)\boldsymbol{v} = -\frac{1}{\rho}\nabla p - \nabla \Phi. \tag{1.2}$$

- Poisson's equation

$$\triangle \Phi = 4\pi G \rho. \tag{1.3}$$

1.2 Linear perturbations and stability analysis

Consider a gas at rest with uniform density and pressure.

Assume that self-gravity does not work when the gas is uniform and isotropic[1]. ($\rho_0 = p_0 =$ const., $\boldsymbol{v}_0 = \Phi_0 = 0$.)

(a) Perturbations: $\rho = \rho_0 + \rho_1, \quad p = p_0 + p_1, \quad \boldsymbol{v} = \boldsymbol{v}_1, \quad \Phi = \Phi_1.$

(b) Perturbation equations (The second-order terms of perturbations are neclected.)

- Eq. of continuity

$$\frac{\partial \rho_1}{\partial t} + \rho_0 \nabla \cdot \boldsymbol{v}_1 = 0. \tag{1.4}$$

- Euler's eq.

$$\frac{\partial \boldsymbol{v}_1}{\partial t} = -\frac{1}{\rho_0}\nabla p_1 - \nabla \Phi_1. \tag{1.5}$$

- Poisson's eq.

$$\triangle \Phi_1 = 4\pi G \rho_1. \tag{1.6}$$

- Eq. of state

$$p_1 = \left(\frac{\partial p}{\partial \rho}\right)_s \rho_1 = c_s^2 \rho_1. \tag{1.7}$$

[1]This assumption of gravitational equilibrium in the unperturbed state is not correct. That is, the gravitational equilibrium would not be reached without the pressure gradient and other effects that balance with gravity. This flaw is referred to as " the Jeans swindle. " Nevertheless, the results of the simple Jeans instability are useful for understanding self-gravitational instabilities in real systems that are in equilibrium with other effects.

(c) Linear solution

Substitution of Eq. (1.6) into [$\frac{\partial}{\partial t}$ Eq. (1.4) $-\rho_0\nabla\cdot$ Eq. (1.5)] yields

$$\frac{\partial^2}{\partial t^2}\rho_1 - c_s^2\triangle\rho_1 - 4\pi G\rho_0\rho_1 = 0. \tag{1.8}$$

The solution of perturbations has a coordinate and time dependence of $\exp(i\boldsymbol{k}\cdot\boldsymbol{x} - i\omega t)$.

Then, we obtain the dispersion relation as

$$\omega^2 = c_s^2 k^2 - 4\pi G\rho_0. \tag{1.9}$$

If ω is imaginary (or complex), the perturbations grow. That is, if

$$k < k_{\rm J} \equiv \frac{\sqrt{4\pi G\rho_0}}{c_s} \quad \longrightarrow \quad \text{unstable} \tag{1.10}$$

- Jeans length $\lambda_{\rm J} = \dfrac{2\pi}{k_{\rm J}} = \sqrt{\dfrac{\pi c_s^2}{G\rho_0}}$ (contraction time $\dfrac{\lambda_{\rm J}}{c_{\rm s}} \sim \dfrac{1}{\sqrt{G\rho_0}}$)
- Jeans mass $M_{\rm J} \simeq \dfrac{4\pi}{3}\rho_0(\lambda_{\rm J}/2)^3$

1.3 Application to contraction of molecular clouds

Molecular Clouds (ex. Orion[1500 lyr], Taurus[450 lyr])

Number density of hydrogen atoms $\sim$ 100-10^4cm^{-3}, $\rho \sim 10^{-22}$-10^{-21}g cm^{-3}.
Size $\sim 10 - 10^3$lyr.
Mass $\sim 10^4$-10^7 M$_\odot$ (M$_\odot = 2\times 10^{33}$g).
Temperature $\sim$ 10-30K (sound velocity $\sim$ 200-300m/s).

- Jeans length $\lambda_{\rm J} \sim 10^{19}$cm $\sim$10lyr.
- Jeans mass $M_{\rm J}$ $\sim$several 10M$_\odot$
- Density dependence of Jeans instability
  $\lambda_{\rm J},\ M_{\rm J} \propto \rho^{-1/2}$ (isothermal)

  The denser the region, the more short-wavelength modes it has, and the more it splits into smaller mass objects.

2 Structures of protoplanetary disks and their self-gravitational instability

Protoplanetary disks form around stars as a byproduct of star formation and are the birthplace of planets.

- Accretion disks
- Passive disks (heated by irradiation from thier central stars) ($\leftrightarrow$ active disks)

We study local self-gravitational stability of gas disks.

2.1 Basic equations in a cylindrical coordinate system

Surface density $\Sigma = \int_{-\infty}^{\infty} \rho dz$, two-dimensional pressure $P = \int_{-\infty}^{\infty} p dz$;
Velocity $v_z = 0$, $\partial \boldsymbol{v}/\partial z \sim 0$

- Eq. of continuity

$$\frac{\partial \Sigma}{\partial t} + \frac{1}{R}\frac{\partial}{\partial R}(R\Sigma v_R) + \frac{1}{R}\frac{\partial}{\partial \phi}(\Sigma v_\phi) = 0 \tag{2.1}$$

- Euler's eq.

$$\frac{\partial v_R}{\partial t} + v_R \frac{\partial v_R}{\partial R} + \frac{v_\phi}{R}\frac{\partial v_R}{\partial \phi} - \frac{v_\phi^2}{R} = -\frac{1}{\Sigma}\frac{\partial P}{\partial R} - \frac{GM_\odot}{R^2} - \frac{\partial \Phi_{\rm D}}{\partial R} \tag{2.2}$$

$$\frac{\partial v_\phi}{\partial t} + v_R \frac{\partial v_\phi}{\partial R} + \frac{v_\phi}{R}\frac{\partial v_\phi}{\partial \phi} + \frac{v_R v_\phi}{R} = -\frac{1}{R\Sigma}\frac{\partial P}{\partial \phi} - \frac{1}{R}\frac{\partial \Phi_{\rm D}}{\partial \phi} \tag{2.3}$$

- Poisson's eq.

$$\triangle \Phi_{\rm D} = 4\pi G \Sigma \delta(z) \tag{2.4}$$

2.2 Vertical hydrostatic structure of gas disks

- For a thin disk, the z-component (vertical component) of Euler's equation is written by

$$\frac{1}{\rho}\frac{\partial p}{\partial z} = -\Omega'^2 z, \tag{2.5}$$

  where the angular frequency, Ω', of vertical oscillation is given by

$$\Omega'^2 = \frac{GM_c}{R^3} + \frac{\partial^2 \Phi_{\rm D}}{\partial z^2}(z=0). \tag{2.6}$$

  If the disk gravity is negligible, Ω' is equal to the Keplerian angular velocity $(GM_c/r^3)^{1/2}$.

- The vertical density profile of the disk can be obtained by solving the vertical hydrostatic equation (2.5). For vertically isothermal cases, we have

$$\rho(z) = \frac{\Sigma}{\sqrt{2\pi}h}\, e^{-z^2/2h^2}. \tag{2.7}$$

  The vertical disk scale height h is given by

$$h = \frac{c_s}{\Omega'}, \tag{2.8}$$

  where c_s is the isothermal sound velocity with $\gamma = 1$. For polytropic disks, we obtain

$$\rho(z) = \rho(0)\left(1 - \frac{(\gamma-1)z^2}{2h^2}\right)^{1/(\gamma-1)}. \tag{2.9}$$

  In this case, h is also given by Eq. (2.8), but c_s in it is evaluated at $z = 0$.

2.3 Gravitational instability of disks

(a) Perturbations: $\Sigma = \Sigma_0 + \Sigma_1, \quad P = P_0 + P_1, \quad P_1 = c_s^2 \Sigma_1;$

$$v_R = v_{R,1}, \quad v_\phi = R\Omega(R) + v_{\phi,1}; \quad \Phi_{\rm D} = \Phi_{\rm D,0} + \Phi_{\rm D,1}$$

From the balance among the centrifugal force, the gravitational forces by the central star and the disk, and the pressure gradient, the angular velocity, Ω, of the unperturbed disk rotation is obtained as

$$\Omega^2 = \frac{GM_c}{R^3} + \frac{1}{R}\frac{\partial \Phi_{\rm D,0}}{\partial R}(z=0) + \frac{1}{R}\frac{\partial H_0}{\partial R}, \tag{2.10}$$

where $H_0 = \frac{\gamma'}{\gamma'-1} P_0/\Sigma_0$ is the unperturbed enthalpy. The angular velocity Ω generally has a radial dependence. In the Keplerian rotation, $\Omega \propto r^{-3/2}$ while $\Omega \propto 1/r$ for galactic disk. A rotation in which Ω depends on r is called a differential rotation, and a rotation in which Ω is independent of r is called a rigid rotation.

(b) WKB approximation for perturbations

$$\frac{\partial}{\partial R} \gg \frac{1}{R}\frac{\partial}{\partial \phi}, \ \frac{1}{R}; \qquad \text{perturbations} \propto \exp(ikR + im\phi - i\omega t)$$

(c) Vertical integration of Poisson's equation

Under the WKB approximation, the perturbation equation of Poisson's equation (2.4) is written as

$$\left(\frac{\partial^2}{\partial r^2} + \frac{\partial^2}{\partial z^2}\right)\Phi_{\rm D,1} = 4\pi G \Sigma_1 \delta(z). \tag{2.11}$$

Integrating this equation from $z = -\epsilon$ to $+\epsilon$ ($\epsilon \ll 1$), and assuming vertical symmetric disks, we obtain

$$\left(\frac{\partial \Phi_{\rm D,1}}{\partial z}\right)_{z=+0} = -\left(\frac{\partial \Phi_{\rm D,1}}{\partial z}\right)_{z=-0} = 2\pi G \Sigma_1. \tag{2.12}$$

Furthermore, since Eq. (2.11) is given by $\left(\frac{\partial^2}{\partial r^2} + \frac{\partial^2}{\partial z^2}\right)\Phi_{\rm D,1} = 0$ for $z \neq 0$, we have

$$\Phi_{\rm D,1} \propto e^{-k|z|} \exp(ikr + im\phi - i\omega t). \tag{2.13}$$

From Eqs. (2.12) and (2.13), we obtain

$$\Phi_{\rm D,1}(z=0) = -2\pi G \Sigma_1/k. \tag{2.14}$$

(d) Other perturbation equations and a dispersion relation

- Eq. of continuity

$$i(m\Omega - \omega)\Sigma_1 + ik\Sigma_0 v_{R,1} + \frac{im\Sigma_0}{R} v_{\phi,1} = 0. \tag{2.15}$$

- Euler's eq.

$$\begin{pmatrix} i(m\Omega - \omega) & -2\Omega \\ -2B & i(m\Omega-\omega) \end{pmatrix} \begin{pmatrix} v_{R,1} \\ v_{\phi,1} \end{pmatrix} = (c_s^2 \Sigma_1/\Sigma_0 + \Phi_{D,1}) \begin{pmatrix} -ik \\ -im/R \end{pmatrix}. \tag{2.16}$$

Solving this yields

$$\begin{pmatrix} v_{R,1} \\ v_{\phi,1} \end{pmatrix} = \frac{c_s^2\Sigma_1/\Sigma_0 + \Phi_{D,1}}{\Delta} \begin{pmatrix} (m\Omega - \omega)k \\ -i2Bk \end{pmatrix}, \tag{2.17}$$

where[2]

$$\begin{cases} B &= -\dfrac{1}{2R}\dfrac{d(R^2\Omega)}{dR} \qquad \text{(Oort's } B \text{ constant)}, \\ \kappa^2 &= -4B\Omega \qquad \text{(epicycle 振動数)}, \\ \Delta &= \kappa^2 - (m\Omega - \omega)^2. \end{cases} \tag{2.18}$$

Substituting Eqs. (2.14) and (2.17) into (2.15), we have

$$i(m\Omega - \omega)\left(1 + \frac{c_s^2 k^2 - 2\pi G\Sigma_0 k}{\Delta}\right)\Sigma_1 = 0. \tag{2.19}$$

Therefore, we obtain the dispersion relation

$$(m\Omega - \omega)^2 = c_s^2 k^2 - 2\pi G\Sigma_0 k + \kappa^2. \tag{2.20}$$

(e) Stability condition
If the frequency ω is real, the perturbation is stable. For real ω, the right-hand side of the dispersion relation must be positive for all k. That is, the stability condition is that "$\kappa^2 > 0$ " and "the discriminant for the right-hand side = 0 is negative" hold. The former requires that the specific angular momentum l $(= r^2\Omega)$ increases with r. It is called the Rayleigh's stability condition for rotating disks. From the latter, we obtain Toomre's stability condition

$$Q \equiv \frac{c_s\kappa}{\pi G\Sigma} > 1, \tag{2.21}$$

where Q is called **Toomre's Q value**. The critical wavelength with $Q = 1$ is given by

$$\lambda_{\rm crit} = 2\pi/k_{\rm crit} = 2\pi c_s/\kappa.$$

Since three frequencies κ, Ω, Ω' are comparable, the critical wavelength is comparable with the disk scale height h. Therefore, Toomre's stability condition may change somewhat due to the effect of the disk thickness. According to the results of a self-gravitational stability analysis for a three-dimensional disk considering the disk thickness, the threshold of Toomre's stability condition (2.21) is estimated to decrease by about 30% (Goldreich & Linden-Bell 1965). However, it is known that the non-axisymmetric perturbation with $m = 2$, which was ignored in the above WKB approximation, becomes unstable and is excited even at $Q \simeq 2$ (Vauterin & Dejonghe 1996).

(f) Derivation of Rayleigh condition based on particle description
We can derive Rayleigh stability condition, $dR^2\Omega/dR > 0$, for disk rotation based on particle description. The energy of an unit-mass particle moving the central force field Φ is given by

$$E = \frac{1}{2}\dot{R}^2 + \frac{1}{2}R^2\Omega^2 + \Phi, \tag{2.22}$$

[2]In Keplerian disks $B = -\Omega/4$, whereas $B = -\Omega$ in rigidly rotating disks. For rigidly rotating disks, the term $-2Bv_{R,1}$ in the ϕ component of Eq. (2.16) is equal to the ϕ component of the Coriolis force.

where $\Omega = \dot{\theta}$. The particle model in Eq. (2.22) can be used to approximate the dynamics of a gas disk rotating at Ω, considering this particle as a single fluid particle.

Let us review the motion of a particle in a central force field. For a particle with the specific angular momentum $j_0 = R_0^2\Omega(R_0)$, Eq. (2.22) can be written as

$$E = \frac{1}{2}\dot{R}^2 + U_{\rm eff}(R), \tag{2.23}$$

where the effective potential $U_{\rm eff}(R)$ is given by

$$U_{\rm eff}(R) = \Phi(R) + \frac{j_0^2}{2R^2}. \tag{2.24}$$

The balance between the central gravity and the centrifugal force on the particle at $R = R_0$ is written as

$$-\frac{dU_{\rm eff}}{dR} = -\frac{d\Phi}{dR} + \frac{j_0^2}{R^3} = 0 \qquad (\text{for } R = R_0). \tag{2.25}$$

We further consider the radial motion of this particle. Differentiating Eq. (2.23) with r, the radial component of the equation of motion is obtained as

$$\ddot{R} = -\frac{dU_{\rm eff}}{dR} = -\frac{d^2U_{\rm eff}}{dR^2}\,(R - R_0). \tag{2.26}$$

In the second equality, we did a Taylor expansion $R = R_0$ and used Eq. (2.25). When $d^2U_{\rm eff}/dR^2 > 0$ (i.e., $U_{\rm eff}$ has a minimum), the solution of Eq. (2.26) is a simple harmonic oscillation around $R = R_0$, and the circular motion of the particle is stable against radial perturbations. Conversely, the case with a negative $d^2U_{\rm eff}/dR^2$ is unstable.

This stability condition can be also expressed in terms of the angular velocity Ω. Noting that $d\Phi/dR = R\Omega^2$ holds at each radius from Eq. (2.25) and differentiating Eq. (2.25), we obtain

$$\frac{d^2U_{\rm eff}}{dR^2} = \frac{dR\Omega^2}{dR} + 3\Omega^2 = \frac{2\Omega}{R}\frac{dR^2\Omega}{dR} = \kappa^2. \tag{2.27}$$

Consequently, the stable condition $d^2U_{\rm eff}/dR^2 > 0$ is equivalent to the Rayleigh condition $dR^2\Omega/dR > 0$. We also find that the epicycle frequency κ is the angular frequency of the radial oscilation.

2.4 Hayashi model for the solar nebular disk and its self-gravitational stability

The Hayashi disk model (or the minimum-mass solar nebula disk) is a standard model of a protoplanetary disk for the formation of the solar system planets.

- Disk temperature (in a thin passive disk without heating sources)

  The gas temperature is equal to the dust temperature which is determined by the balance between the heating by the solar radiation and the cooling by the thermal emission from dust grains. For a dust grain with a radius d, this energy balance is written as

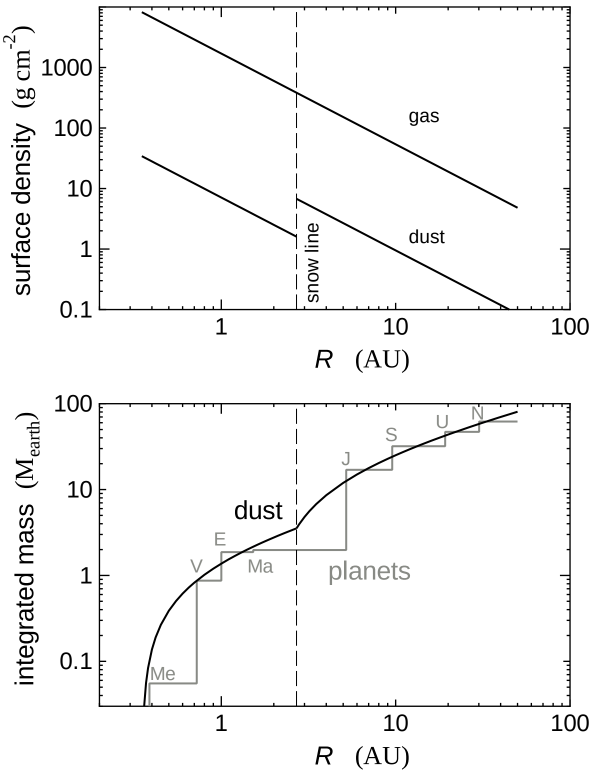

Figure 1: Distributions of Gas and dust surface densities in the Hayashi model (upper panel). The vertical dashed line indicates the location of the snow line. The lower panel shows the integrated dust mass. The inner radius of the disk is 0.35AU. The gray line shows the cumulative solid mass of the planets in the solar system. The solid masses of each giant planets are set to be $15M_\oplus$ as in Hayashi (1981).

$$\pi d^2 L_\odot/(4\pi R^2) = 4\pi d^2 \sigma T^4, \tag{2.28}$$

where σ is the Stefan-Boltzmann constant and R is the distance to the star. The stellar luminosity is set to be the solar value ($L_\odot = 3.83\times10^{33}$erg/sec). From the above equation, the disk temperature and the isothermal sound speed are evaluated as

$$T = \left(\frac{L_\odot}{16\pi\sigma R^2}\right)^{1/4} = 280(R/1\mathrm{AU})^{-1/2}\,\mathrm{K}, \tag{2.29}$$

$$c_s = 1.2(R/1\mathrm{AU})^{-1/4}\,\mathrm{km/sec}, \tag{2.30}$$

where the mean molecular weight is set to be 2.3 and the heat capacity ratio is 1.4.

- Disk surface density (the minimum value for formation of the planets)

$$\Sigma_{\rm gas} = 1700(R/1\mathrm{AU})^{-3/2}\ \mathrm{g/cm^2}, \tag{2.31}$$

$$\Sigma_{\rm dust} = \begin{cases} 7\,(R/1\mathrm{AU})^{-3/2}\ \mathrm{g/cm^2} & (R < 2.7\mathrm{AU},\ \text{silicate dust}), \\ 28\,(R/1\mathrm{AU})^{-3/2}\ \mathrm{g/cm^2} & (R > 2.7\mathrm{AU},\ \text{silicate \& ice}). \end{cases} \tag{2.32}$$

- Mass of the gas disk: $\quad M_{\rm disk} = \int_0^{50AU} \Sigma_{\rm gas}\, 2\pi R dR \simeq 0.017 M_\odot$

  Dust mass $\simeq 2.3 \times 10^{-4} M_\odot$ (about 80 Earth masses)

- Toomre's Q-value

$$Q = 66\,(R/1{\rm AU})^{-1/4} \tag{2.33}$$

  Therefore, the Hayashi model disk is stable against self-gravitational instability. Note that the Q-value decreases outside the disk. A more massive disk can become gravitationally unstable in the outer part.

- Disk thichness

  The scale height of the gas disk is given by $h = c_s/\Omega$, where c_s is the isothermal sound velocity with $\gamma = 1$. The disk aspect ratio h/R is $\simeq 1/30(R/1AU)^{1/4}$.

Problem 1.

1. - Calculate the Jeans length and the Jeans mass for a molecular cloud with the hydrogen number density of 100 cm^{-3} and temperature of $10K$.
   - Derive the temperature of Eq. (2.30) the sound velocity of Eq. (2.30) for the Hayashi model disk with the mean molecular weight of 2.3. Also calculate Toomre's Q-value for the Hayashi model disk, assuming that $\kappa = \Omega$.

2. Derive Eqs. (2.17) and (2.19).

3. The Roche limit (radius) is given by

$$R_{Roche} \simeq 2.4 R_M (\rho_M/\rho_m)^{1/3}, \tag{2.34}$$

   where R_M and ρ_M are the radius and the average density of the central star, respectively; and ρ_m is the average density of the secondary orbiting the central star. If the secondary orbits its central star outside the Roche limit, it can avoid tidal disruption due to its self-gravity. Show that this condition is similar to Toomre's stability condition. (Hint: The mass of the central star is $M = \frac{4\pi}{3}\rho_M R_M^3$. The disk density is related to the surface density with $\rho \sim \Sigma/h$.)

4. The Q-value of the galactic disk in the vicinity of the Sun: Near the Sun (at 8 kpc from the galactic center), the sum of the densities of stars and interstellar gas is estimated to be ~ 0.1 M$_\odot$ pc^{-3}. The surface density Σ of the galactic disk can be roughly estimated by multiplying this density by the thickness h. Assuming the rotation speed of the galactic disk to be 200 km/s and approximating $\kappa = \Omega'$, we can estimate the Q-value of the galactic disk in the vicinity of the Sun (the solar mass is 2×10^{30}kg, 1pc $= 3 \times 10^{16}$m). The result will be $Q \sim 1$. This is consistent with the theory that the spiral structure of the galactic disk is formed by gravitational instability.

3 Evolution and structure of accretion disk

Accretion disks around black holes and protoplanetary disks evolve due to viscosity. Here we describe the evolution of viscous accretion disks. An accretion disk rotates around its host star at approximately Keplerian angular velocity. It is a differential rotation, where the inner part rotates rapidly and the outer part rotates slowly. When viscosity acts on a differentially rotating disk, the rapidly rotating inner disk material experiences a negative torque from the outer material and slows down. Thus, viscosity transfers angular momentum from the inner disk to the outer disk. The inner disk loses the angular momentum and falls inward, while the outer disk expands outward. This results in mass accretion onto the host star, reducing the disk mass and increasing the disk radius.

3.1 Basic equations for accretion disks

We examine the evolution of the viscous accretion disk in detail using the hydrodynamical equations with the addition of viscous effects. We consider an axi-symmetric disk, which is assumed here to be a two-dimensional disk. To describe the disk, we use the polar coordinate system (r, ϕ) with the host star at the origin. For two-dimensional axi-symmetric disks, the equation of continuity (2.1) is rewritten as

$$\frac{\partial \Sigma}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}\left(r\Sigma v_r\right) = 0. \tag{3.1}$$

We do not consider any inflow onto the disk or outflow except the accretion onto the host star. For accretion disks, we use the Navier-Stokes equation with the viscosity term instead of the Euler equation. Around a host star, the Navier-Stokes equation is given by

$$\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v}\cdot\mathbf{grad})\boldsymbol{v} = -\frac{1}{\rho}\mathbf{grad}\, p + \mathbf{grad}\left(\frac{GM_c}{r}\right) + \frac{1}{\rho}\mathrm{div}\,\boldsymbol{\Pi'} \tag{3.2}$$

where Π' is the viscous stress tensor given by[3]

$$\Pi'_{ij} = \rho\nu\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right) \tag{3.3}$$

where ν is the kinetic viscosity. We do not consider any external forces other than the gravity of the host star. For two-dimensional axi-symmetric disks, the ϕ-component of the Navier-Stokes equation is rewritten as[4]

$$\frac{\partial v_\phi}{\partial t} + v_r\frac{\partial v_\phi}{\partial r} + \frac{v_r v_\phi}{r} = \frac{1}{\Sigma}\left(\frac{1}{r}\frac{\partial\, r\Pi'_{r\phi}}{\partial r} + \frac{\Pi'_{r\phi}}{r}\right) \tag{3.4}$$

and the r,ϕ-component of the viscous stress, $\Pi'_{r\phi}$, is given for two-dimensional disks by

$$\Pi'_{r\phi} = \Sigma\nu\left(\frac{\partial v_\phi}{\partial r} - \frac{v_\phi}{r}\right) = \Sigma\nu r\frac{d\Omega}{dr}. \tag{3.5}$$

[3]For accretion disks, the gas velocity in the frame rotating with the disk is smaller than the sound speed and we can assume the incompressible fluid.

[4]In the polar coordinate system, the term of $(\boldsymbol{v}\cdot\mathbf{grad})\boldsymbol{v}$ has additional terms $-\boldsymbol{e}_R v_\phi^2/R + \boldsymbol{e}_\phi v_R v_\phi/R$ and the latter one appears in the equation (3.4). Since the velocity vector is expressed as $\boldsymbol{v} = v_R\boldsymbol{e}_R + v_\phi\boldsymbol{e}_\phi$, the gradient operates on the basis vectors as well as the velocity components. Noting this, and using $\partial\boldsymbol{e}_R/\partial\phi = \boldsymbol{e}_\phi$ and $\partial\boldsymbol{e}_\phi/\partial\phi = -\boldsymbol{e}_R$, we can obtain the additional terms above. The additional term of the viscosity term in the equation (3.4) and that of the viscous stress in the equation (3.5) are also derived in the same way.

By the definition of the viscous stress, $\Pi'_{r\phi}(r)$ represents the ϕ component of the force (per unit length) exerted on the inner disk material by the outer material tangent to the inner material at R. The angular velocity of the disk, Ω, is determined by the balance mainly between the stellar gravity and centrifugal force in the radial component of the equation (3.2) and is approximately given by $\Omega_{\rm K}$. As seen in the equation (3.5), the sign of the viscous stress $\Pi'_{R\phi}$ is determined by the gradient of Ω. In a uniformly rotating disk with constant Ω, the viscous stress does not work. In Keplerian disks with $\Omega_{\rm K}$, the negative viscous torque is exerted on the inner material by the outer material.

The angular momentum conservation equation for accretion disks is obtained from the equations (3.1), (3.4), and (3.5) as

$$\frac{\partial}{\partial t}\left(\Sigma\, j\right) + \frac{1}{r}\frac{\partial}{\partial r}\left(r\Sigma\, j v_r - r^3\Sigma\nu\frac{d\Omega}{dr}\right) = 0, \tag{3.6}$$

where $j \,(= R^2\Omega)$ is the specific angular momentum. The second term in the left-hand side of the equation (3.6) is the divergence of the radial angular momentum flux (density). The first term of the angular momentum flux shows the flux due to advection and the second is that due to the viscous torque. Using the equations (3.1) and (3.6), and noting that $\partial j/\partial t = 0$, we also obtain v_R and the inward mass flux (i.e., the accretion rate) of the disk as

$$\dot{\mathcal{M}} \equiv -2\pi r\Sigma\, v_r = -\frac{2\pi}{(dj/dr)}\frac{\partial}{\partial r}\left(r^3\Sigma\,\nu\frac{d\Omega}{dr}\right). \tag{3.7}$$

Substituting this into the equation (3.1), we finally obtain the equation describing the viscous evolution of accretion disks as

$$\frac{\partial\Sigma}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}\left[\frac{1}{(dj/dr)}\frac{\partial}{\partial r}\left(r^3\Sigma\nu\frac{d\Omega}{dr}\right)\right] = 0. \tag{3.8}$$

For a disk in Keplerian rotation with $\Omega = \Omega_{\rm K}$, it is reduced to

$$\frac{\partial\Sigma}{\partial t} - \frac{3}{r}\frac{\partial}{\partial r}\left[r^{1/2}\frac{\partial}{\partial r}\left(r^{1/2}\nu\Sigma\right)\right] = 0. \tag{3.9}$$

The viscosity of accretion disks is determined by the turbulent viscosity, not the molecular viscosity. The origin of the turbulence has not yet been determined for protoplanetary disks, although magneto-rotational instability and self-gravitational instability are strong candidates, and the magnitude of the viscosity has a large uncertainty. Therefore the kinetic viscosity of accretion disks is often expressed in terms of a non-dimensional parameter as

$$\nu = \alpha h^2\Omega. \tag{3.10}$$

This simple expression for the viscosity is known as the Shakura-Sunyaev α prescription and α is called Shakura-Sunyaev α parameter (Shakura & Sunyaev, 1973).

3.2 Solution for steady-state accretion disks

We consider a steady solution for (inward) accreting disks. Setting $\partial/\partial t = 0$ in the equations (3.8) and (3.6), we obtain the mass and angular momentum fluxes as

$$\dot{\mathcal{M}} = \text{const.}, \tag{3.11}$$

$$\dot{\mathcal{J}} \equiv j\dot{\mathcal{M}} + 2\pi r^3\Sigma\nu\frac{d\Omega}{dr} = \text{const.} \tag{3.12}$$

Note that inward fluxes are defined to be positive. For accretion disks, thus, $\dot{\mathcal{M}}$ is positive. Furthermore, assuming $\Sigma = 0$ at the inner disk edge $r_{\rm in}$ as the inner boundary condition, we obtain

$$\dot{\mathcal{J}} = \dot{\mathcal{M}} j(r_{\rm in}). \tag{3.13}$$

Thus the angular momentum flux is negative in steady accretion disks. Substituting this into the equation (3.12), we obtain the steady surface density as

$$\Sigma = -\dot{\mathcal{M}} \frac{j(r) - j(r_{\rm in})}{2\pi\nu r^3 (d\Omega/dr)}. \tag{3.14}$$

For Keplerian disks, it is rewritten as

$$\Sigma = \frac{\dot{\mathcal{M}}}{3\pi\nu}\left(1 - \sqrt{\frac{r_{\rm in}}{r}}\right). \tag{3.15}$$

3.3 Similarity solution for accretion disks

Next we consider a time-evolving solution to equation (3.9). Suppose that the kinematic viscosity is given by a power-law function

$$\nu = \nu_0 R^\gamma. \tag{3.16}$$

In this case, it is known that there exists a similarity solution (Lynden-Bell & Pringle 1974). In the similarity solution for an accretion disk, the disk radius and surface density evolve with time, but the surface density distribution remains in a similar form.

(a) Dimensional analysis for evolution of accretion disks

The time evolution of the similarity solution can be clarified by dimensional analysis. An accretion disk spreads due to viscosity and its radius $R_{\rm d}$ increases. Suppose that the disk is formed at $t = 0$ in a small size and then spreads out due to the viscous effect. Since the equation (3.9) is a second-order differential equation for space and has the form of a diffusion equation, time t is approximately equal to the viscous diffusion time of the disk, $R_d^2/\nu(R_d)$. Then, also using the equation (3.16), the radius of the disk is approximately given by a power-law function of time as

$$R_d \simeq (\nu_0 t)^{\frac{1}{2-\gamma}}. \tag{3.17}$$

Note that ν_0 does not have the dimension of the diffusion coefficient [cm^{2}s^{-1}].

In the similarity solution, the inner edge radius $R_{\rm in}$ is assumed to be much smaller than the disk radius. Since the angular momentum flux at the inner edge is also negligibly small, the total angular momentum, $J_{\rm d}$, of the disk is conserved. We can use the constant $J_{\rm d}$ to estimate the evolution of the disk mass. Estimating the characteristic value of the specific angular momentum of the disk as $R_d^2\,\Omega(R_d)$, the disk mass $M_{\rm d}$ is approximately given by

$$M_{\rm d} \simeq \frac{J_{\rm d}}{R_d^2\,\Omega(R_d)} \quad \propto t^{-\frac{1}{2(2-\gamma)}}. \tag{3.18}$$

In the above, the time dependence is derived using $\Omega \propto R^{-3/2}$. Furthermore, the characteristic value of the disk surface density can be estimated as

$$\Sigma(R_{\rm d}(t), t) \simeq \frac{J_{\rm d}}{R_d^4\,\Omega(R_d)} \quad \propto t^{-\frac{5}{2(2-\gamma)}}. \tag{3.19}$$

The similarity solution for the surface density also depend on a non-dimensional "similarity" variable, $y = r^{2-\gamma}/(\nu_0 t) \simeq (r/R_\mathrm{d})^{2-\gamma}$. The surface density distribution of the disk is determined by its y-dependence.

(b) Exact form of the similarity solution

The similarity solution to (3.9) is written as (Lynden-Bell & Pringle 1974; Hartmann et al. 1998; see also Appendix for the derivation)

$$\Sigma(r,t) = \frac{|\dot{M}_\mathrm{d}(t)|}{3\pi\nu} \exp\left[-\left(\frac{r}{R_d(t)}\right)^{2-\gamma}\right], \tag{3.20}$$

The disk radius R_d is given by

$$R_d = \left[3(2-\gamma)^2 \nu_0 t\right]^{\frac{1}{2-\gamma}} \tag{3.21}$$

and the disk mass M_d and its time derivative are

$$M_\mathrm{d} = \frac{J_\mathrm{d}}{\Gamma(b)\, R_\mathrm{d}^2 \Omega(R_\mathrm{d})}, \qquad \dot{M}_\mathrm{d} = -\frac{M_\mathrm{d}}{2(2-\gamma)t}, \tag{3.22}$$

where $b = (5-2\gamma)/(4-2\gamma)$ and $\Gamma(b)$ is the Gamma function. We can see that these expressions of the similarity solution are consistent with the above estimates by dimensional analysis. We also find that the similarity solution (3.20) agrees with the steady solution (3.14) in the radial range of $r_\mathrm{in} \ll r \ll R_\mathrm{d}$. In this range Σ is proportional to $1/\nu$ or $r^{-\gamma}$, and it is exponentially truncated at a radius R_d. The inward mass and angular momentum fluxes are written as

$$\dot{\mathcal{M}} = -2\pi r \Sigma v_r = 3\pi\nu\Sigma \left[1 - 2(2-\gamma)\left(\frac{r}{R_d}\right)^{2-\gamma}\right], \tag{3.23}$$

$$\dot{\mathcal{J}} = -6\pi(2-\gamma) j\nu\Sigma \left(\frac{r}{R_d}\right)^{2-\gamma}, \tag{3.24}$$

respectively. The angular momentum is always transferred outward in the similarity solution[5]. The equation (3.23) also gives the radial velocity.

We estimate the life time of protoplanetary disks using the similarity solution. Adapting the α prescription for the viscosity (equation [3.10]) and assuming a constant α and $T \propto r^{1/2}$, we obtain $\nu \propto r$ and $\gamma = 1$. The disk life time is approximately given by

$$t_\mathrm{d} \simeq \frac{R_\mathrm{d}^2}{3(2-\gamma)^2\nu} = \frac{R_\mathrm{d}^2}{3\alpha h^2 \Omega(R_\mathrm{d})}. \tag{3.25}$$

If $\alpha = 10^{-3}$, the life time of a protoplanetary disk is estimated to be 5 Myr for the disk with the radius of 100au (and $h/R_d \simeq 0.1$), which is almost consistent with the observed life time of protoplanetary disks. Thus we expect that $\alpha = 10^{-3}$ might be the typical value for protoplanetary disks. The second equation of (3.22) gives a simple relation between the mass accretion rate and the disk mass. For a 1Myr old disk with the mass of $0.02\mathrm{M}_\odot$, the mass accretion rate is obtained as $10^{-8}\mathrm{M}_\odot/\mathrm{yr}$, which is the typical value of the observed accretion rate.

[5]Note that steady accretion disks have a small inward angular momentum flux (see eq. [7.41]). A realistic accretion disk with a finite inner radius r_in also has an inward angular momentum flux in the innermost part, even though the angular momentum flux is outward in the most of the rest of the disk.

(c) Derivation of the similarity solution

We briefly describe the derivation of the similarity solution. From the given parameters, ν_0 and J_d, and two independent variables, r and t, we can form only one dimensionless variable, which can be written as

$$y = \frac{r^2}{\nu t} = \frac{r^{2-\gamma}}{\nu_0 t}. \tag{3.26}$$

The time and radial dependences of the similarity solution are described only by this dimensionless variable y. The radial distribution of the angular momentum spreads out with the increase in the disk radius, but its distribution in the y-space does not change due to the similarity. That is, the angular momentum of the disk inside a radius $r(y)$ that changes so that $y =$constant, must remain constant. It is easy to write down the equation that expresses the constancy of this angular momentum. The angular momentum flowing out of a radius r in unit time is given by the angular momentum flux, $-\dot{\mathcal{J}}$, which is defined by the equation (3.12). On the other hand, since the radius r corresponding to a given y increases with the velocity of $\frac{dr(y)}{dt}$, an area inside the radius $r(y)$ increases in unit time by $2\pi r\frac{dr(y)}{dt}$, and the angular momentum in this additional area is $j\Sigma\, 2\pi r\frac{dr(y)}{dt}$. These angular momenta equals due to constancy of the angular momentum distribution in the y-space, and we obtain

$$\dot{\mathcal{M}} \equiv -2\pi r\Sigma v_r = 2\pi\beta\nu\Sigma\left[1 - \frac{y}{\beta(2-\gamma)}\right]. \tag{3.27}$$

Using Eqs. (3.21) and (3.26), we can see that this is equal to Eq. (3.23).

The solution of the surface density can be expressed as

$$\Sigma = \frac{J_d}{r^4\Omega(r)}f(y), \tag{3.28}$$

where $f(y)$ is a dimensionless function and the prefactor has the dimension of a surface density. Substituting these expressions for Σ and v_r into the equation (3.7), we obtain a differential equation for f as $d\ln f/d\ln y = -y/[3(2-\gamma)^2] + b$. Solving this equation yields the solution

$$f = \frac{2-\gamma}{2\pi\Gamma(b)}x^b\exp(-x) \tag{3.29}$$

with the new variable $x = y/[3(2-\gamma)^2]$, and also gives Σ. In the equation (3.29), the coefficient is determined by the condition that the total angular momentum calculated with Σ should equal J_d. Using this solution of Σ, we obtain the disk mass as the equation (3.22). Finally, using the equation (3.22), the solution of Σ is rewritten as the equation (3.20).

Problem 2. Show that $dr(y)/dt = \nu y/[(2-\gamma)r]$ and derive Eq. (3.27). Also derive the differential equation for $f(y)$ and its solution (3.29). Then, derive Eqs. (3.22) and (3.20), too.

Problem 3. The obtained similarity solution (3.20) is physically meaningless when the power-law index γ of the viscosity is larger than 2. Find the physical reason why $\gamma < 2$ is required for the similarity solution by explaining how the physical property of the disk evolution changes between the cases with $\gamma < 2$ and $\gamma > 2$.

(ASIDE) A similarity solution had also been derived for accretion disks where the viscosity also depends on the surface density as $\nu = \nu(r, \Sigma) = \nu_0 r^\gamma \Sigma^\delta$ (Pringle 1974, 1991; Cannizzo et al. 1990). Let us obtain such a similarity solution in the same way as above. Here we consider Keplerian disks with $\Omega = \sqrt{GM_*/r^3}$.

Using the characteristic disk radius R_c, the characteristic value of the surface density Σ_c is given by $\Sigma_c = J_d/[R_c^4\Omega(R_c)]$. The characteristic disk radius R_c also satisfies $R_c^2 = t\nu(R_c, \Sigma_c)$. Then, we obtain $R_c(t) = [(J_d/\sqrt{GM_*})^\delta \nu_0 t]^{1/a}$, where $a = 2 - \gamma + \frac{5}{2}\delta$ (cf. eq. [3.17]).

In this case, the dimensionless variable y is defined by $y \equiv r^2/[t\nu(r, \Sigma_r)] = [r/R_c(t)]^a$, where $\Sigma_r = J_d/[r^4\Omega(r)]$ (cf. eq. [3.26]). Solving this difinition of y for r yields $r(y) = R_c(t)y^{1/a}$. Then, $dr(y)/dt$ and the inward mass flux are given by

$$\frac{dr(y)}{dt} = \frac{\nu(r, \Sigma_r)y}{ar}, \qquad \dot{\mathcal{M}} = 3\pi\nu\Sigma\left[1 - \frac{2y}{3a}\left(\frac{\Sigma}{\Sigma_r}\right)^{-\delta}\right]. \tag{3.30}$$

The derivation is similar to that of Eq. (3.27).

The similarity solution is written as $\Sigma = \Sigma_r f(y)$. From the equivalence of Eqs. (3.7) and (3.30), we obtain the differential equation for $f(y)$, $\frac{df^\delta}{dy} - A\frac{f^\delta}{y} + B = 0$, where $A = \frac{1+\frac{1}{2a}}{1+\frac{1}{\delta}}$ and $B = \frac{1}{3a^2(1+\frac{1}{\delta})}$. Solving this differential equation, we obtain the solution for the disk surface density with the total angular momentum J_d as

$$\Sigma = C\,\frac{J_d}{r^4\Omega}(x^A - x)^{1/\delta}, \quad x = \frac{By}{(1-A)C^\delta}, \quad C = \left[\frac{2\pi}{a}\int_0^1 (x^A - x)^{1/\delta}\,\frac{dx}{x}\right]^{-1}. \tag{3.31}$$

3.4 Disk heating by viscous dissipation

The dissipation energy due to viscosity per unit volume per unit time, ϵ, is given for axi-symmetric Keplerian disks by (e.g., Landau & Lifshitz 1959, "Fluid Mechanics")

$$\epsilon = \Pi'_{ik}\frac{\partial v_i}{\partial x_k} = \rho\nu\left(r\frac{d\Omega}{dr}\right)^2 = \frac{9}{4}\rho\nu\Omega^2. \tag{3.32}$$

Vertical integration gives the heating rate per unit area of the disk. It is balanced by the radiative cooling rate at the upper and lower disk surfaces given by $2\sigma T_s^4$, where T_s is the temperature at the disk surface. Furthermore, assuming a steady accretion disk, the surface temperature is obtained as

$$T_s = \left(\frac{3GM_*\dot{\mathcal{M}}}{8\pi\sigma r^3}\right)^{1/4} \quad \propto\ r^{-3/4}. \tag{3.33}$$

It is assumed above that $r \gg r_{\rm in}$. The mass accretion rate of $\dot{\mathcal{M}} = 10^{-8}\mathrm{M}_\odot/\mathrm{yr}$ gives $T_s = 90\mathrm{K}$ at 1au. This is lower than the temperature of the Hayashi model for protoplanetary disks, which is heated by the stellar radiation[6]. Since T_s has a steeper radial gradient than that of the Hayashi model, the viscous heating can dominate the stellar radiation heating at an inner radius with $R \ll 1\mathrm{au}$.

[6] The ratio between two temperatures given by the equations (3.33) and (2.29) is determined by the ratio of the viscous heating rate to that by the stellar radiation, which given by

$$\frac{6GM_*\dot{\mathcal{M}}/r}{L_*} \simeq 10^{-2}\left(\frac{\dot{\mathcal{M}}}{10^{-8}\mathrm{M}_\odot/\mathrm{yr}}\right)\left(\frac{M_*}{M_\odot}\right)\left(\frac{L_*}{L_\odot}\right)^{-1}\left(\frac{r}{1\mathrm{au}}\right)^{-1}.$$

Optically thick protoplanetary disks can have an inner temperature much higher than T_s. When the energy dissipation due to viscosity is concentrated to the disk midplane or uniformly distributed (i.e., the dissipation rate $\propto \rho$), the midplane temperature is approximately given by $\sim \tau^{1/4} T_s$, where the vertical optical depth τ is defined by $\kappa_R \Sigma$ and κ_R is the Rosseland mean opacity of the disk material. However, if the viscous heating occurs only at the disk surface, the midplane temperature is similar to T_s (Mori et al. 2019).

We also describe the energy balance at each radius of viscous accretion disks. Taking the scalar product of the equation (3.2) with $\boldsymbol{v}$, we obtain the equation for the kinetic energy per unit mass as

$$\begin{aligned}\left[\frac{\partial}{\partial t} + (\boldsymbol{v} \cdot \mathbf{grad})\right]\left(\frac{v^2}{2}\right) = & -\frac{1}{\rho}\boldsymbol{v} \cdot \mathbf{grad}\, p - \boldsymbol{v} \cdot \mathbf{grad}\left(-\frac{GM_*}{r}\right) \\ & + \frac{1}{\rho}\mathrm{div}\,(\boldsymbol{v} \cdot \boldsymbol{\Pi'}) - \frac{1}{\rho}\frac{\partial v_i}{\partial x_k}\Pi'_{ik}. \end{aligned} \tag{3.34}$$

In the above equation, the term of the pressure gradient can be neglected since $c_s^2 \sim p/\rho$ is much smaller than the rotational energy of $v_\phi^2/2 = r^2\Omega^2/2$ for standard thin accretion disks. Furthermore, since $|v_R| \ll |v_\phi|$, the kinetic energy of $v^2/2$ is replaced by $v_\phi^2/2$ and only $\Pi'_{r\phi}$ should be considered for the viscous stress tensor. Thus the equation (3.34) can be rewritten for two-dimensional disk as

$$-\dot{\mathcal{M}}\frac{\partial}{\partial r}\left(\frac{r^2\Omega^2}{2}\right) = \dot{\mathcal{M}}\frac{\partial}{\partial r}\left(-\frac{GM_*}{r}\right) - \frac{\partial}{\partial r}\left(3\pi\Sigma\nu r^2\Omega^2\right) - 2\pi r\Sigma\,\frac{9}{4}\nu\Omega^2. \tag{3.35}$$

This equation describes the energy balance at each radius of steady viscous accretion disks.

Problem 4. The total viscous heating rate of an entire steady-state Keplerian accretion disk around a central star with mass M_* is expected to be given by $1/2(GM_*/R_{\mathrm{in}})\dot{M}_d$, since it is supplied by the release of gravitational energy. Check that this prediction is valid by integrating Eq. (3.32 over the entire disk. Assume that the inner edge of the disk, R_{in}, is sufficiently smaller than the disk radius.

Problem 5. Derive Eq. (3.35) and explain the physical meaning of four terms in this equation. Furthermore, caluculate the ratios between these four terms and explain the energy balance at each radius of steady viscous accretion disks.

3.5 Disk dissipation

- Observational constraints on disk dispersion
  - Disk lifetime $\sim 10^6$-10^7yr.
  - The relatively small number of disks in the dissipation process (i.e., transition disks) indicates that the dissipation is rapid ($\sim 10^5$yr).

  A dissipation mechanism other than gradual viscous evolution is required. $\longrightarrow$ "Photo-evaporation" is promising

- Disk dissipation due to photoevaporation

– FUV, EUV, and/or X-rays from the central star (or other stars) heat up the disk surface ($\sim 10^4$K), which causes hydrodynamic escape.

– Photoevaporation occurs outside the critical radius r_g.

$$r_g = \frac{GM_*}{c_s^2} = 9\left(\frac{M_*}{M_\odot}\right)\left(\frac{c_s}{10\mathrm{km/s}}\right)^{-2} \mathrm{AU}. \tag{3.36}$$

– The disk evaporation rate due to photoevaporation is $\dot{M}_w = 10^{-9}$-$10^{-7}M_\odot/\mathrm{yr}$, depending the intensity of FUV, EUV, and/or X-rays.

4 Motion of Dust Particles and Dust Growth in Protoplanetary Disks

- The initial state of dust is interstellar dust. The size distribution of interstellar dust particles ranges from 5nm to 0.2μm, but most of the mass is carried by the upper size limit (Mathis et al. 1977). Each of initial dust particles has a silicate core and an ice mantle. They grow via sticking.
- As the turbulence in the planetary disk weakens, the dust particles settle to the mid-plane of the disk, forming a dense dust layer.
  $\rightarrow$ Gravitational instability of the dust layer leads to the formation of planetesimals (Goldreich and Ward 1973).
- Small dust particles cannot settle due to even slight turbulent gas motion.
  $\rightarrow$ Dust growth is important.

4.1 Motion of Dust Particles

(a) Gas drag on dust particles

- Gas drag force

$$\boldsymbol{F}_{\mathrm{drag}} = -mA(m)\rho_{\mathrm{g}}\Delta\boldsymbol{v}, \tag{4.1}$$

where the coefficient A is given by the two expressions, depending on the size.

– Stokes' law $\quad (a > 9l/4, \quad l\,(= 1/[n_{H_2}\sigma_{H_2}])$: gas mean free path)

$$A = \frac{9v_{\mathrm{th}}l}{4\rho_{\mathrm{solid}}a^2} = \sqrt{\frac{8}{\pi}}\frac{9c_{\mathrm{s}}l}{4\rho_{\mathrm{solid}}a^2}, \tag{4.2}$$

where $v_{\mathrm{th}} = \sqrt{8/\pi}c_s, \quad \boldsymbol{F}_{\mathrm{drag}} = -6\pi\nu a\rho_{\mathrm{g}}\Delta\boldsymbol{v}, \quad \nu = v_{\mathrm{th}}l/2.$

– Epstein's law $\quad (a < 9l/4)$

$$A = \frac{v_{\mathrm{th}}}{\rho_{\mathrm{solid}}a} = \sqrt{\frac{8}{\pi}}\frac{c_{\mathrm{s}}}{\rho_{\mathrm{solid}}a}. \tag{4.3}$$

– General relation between A and C_D

$$A = \frac{3\,C_D\Delta v}{8\rho_{\mathrm{solid}}a}. \qquad \text{(For high Reynolds numbers or supersonic flows, } C_D \sim 1\text{)} \tag{4.4}$$

- Stopping time

$$t_{\rm stop} = \frac{m\Delta v}{|\boldsymbol{F}_{\rm drag}|} = \frac{1}{A\rho_{\rm g}}, \qquad \boldsymbol{F}_{\rm drag} = -\frac{m\Delta \boldsymbol{v}}{t_{\rm stop}}. \tag{4.5}$$

(b) Eqs. of motion for gas and dust (for steady, axisymetric, and non-self-gravitating disks)

- Gas velocity: $\boldsymbol{v} = (v_R, v_\phi = R\Omega + v_{\phi,1}, v_z)$. ($\Omega = \Omega_{\rm K} \equiv \sqrt{GM_{\rm c}/R^3}$. $B = -\Omega/4$.)

  From Eq. (2.16), we have

$$\begin{pmatrix} 0 & -2\Omega \\ -2B & 0 \end{pmatrix} \begin{pmatrix} v_R \\ v_{\phi,1} \end{pmatrix} = \begin{pmatrix} -\dfrac{1}{\rho_{\rm g}}\dfrac{\partial p}{\partial R} \\ 0 \end{pmatrix} + \rho_{\rm d} A \begin{pmatrix} V_R - v_R \\ V_{\phi,1} - v_{\phi,1} \end{pmatrix}. \tag{4.6}$$

- Dust velocity: $\boldsymbol{V} = (V_R, V_\phi = R\Omega + V_{\phi,1}, V_z)$

$$\begin{pmatrix} 0 & -2\Omega \\ -2B & 0 \end{pmatrix} \begin{pmatrix} V_R \\ V_{\phi,1} \end{pmatrix} = \qquad - \rho_{\rm g} A \begin{pmatrix} V_R - v_R \\ V_{\phi,1} - v_{\phi,1} \end{pmatrix}. \tag{4.7}$$

(c) Solution in the case where $\rho_{\rm d} \ll \rho_{\rm g}$

- Gas

$$v_R = v_z = 0, \qquad v_\phi = (1-\eta) R\Omega, \tag{4.8}$$

$$\eta = -\frac{1}{2R\Omega^2\rho_{\rm g}}\frac{\partial p}{\partial R} = -\frac{1}{2}\left(\frac{c_{\rm s}}{R\Omega}\right)^2 \frac{\partial \ln p}{\partial \ln R}. \tag{4.9}$$

- Dust

$$V_R = -\frac{2t_{\rm stop}\Omega}{1+(t_{\rm stop}\Omega)^2}\,\eta R\Omega, \tag{4.10}$$

$$V_\phi - v_\phi = \frac{(t_{\rm stop}\Omega)^2}{1+(t_{\rm stop}\Omega)^2}\,\eta R\Omega, \tag{4.11}$$

$$V_z = -\,\Omega z\, t_{\rm stop}\Omega. \tag{4.12}$$

(d) Solution for arbitrary gas-to-dust ratio $\epsilon = \rho_{\rm d}/\rho_{\rm g}$ (Nakagawa et al. 1986)

$$\begin{pmatrix} v_R \\ v_{\phi,1} + \eta R\Omega \end{pmatrix} = \frac{\eta R\Omega}{(1+\epsilon)^2 + (t_{\rm stop}\Omega)^2} \begin{pmatrix} 2\,\epsilon\, t_{\rm stop}\Omega \\ \epsilon(1+\epsilon) \end{pmatrix}. \tag{4.13}$$

$$\begin{pmatrix} V_R \\ V_\phi - v_\phi \end{pmatrix} = \frac{\eta R\Omega}{(1+\epsilon)^2 + (t_{\rm stop}\Omega)^2} \begin{pmatrix} -2\, t_{\rm stop}\Omega \\ (t_{\rm stop}\Omega)^2 \end{pmatrix}. \tag{4.14}$$

4.2 Dust growth and settling

(a) Dust growth rate

$$\frac{dm}{dt} = \rho_{\rm d}\, \sigma_{\rm col}\, v, \tag{4.15}$$

where the collision cross section $\sigma_{\rm col}$ is equal to $4\pi a^2$ and the collison velocity v is assumed to be comparable to the settling velocity V_z.

The growth time of dust particles is

$$t_{\rm grow} = a/\frac{da}{dt} = 3m/\frac{dm}{dt} = \frac{3m\rho_{\rm g} A}{4\pi a^2 \rho_{\rm d} z \Omega^2} \sim \frac{\Sigma_{\rm g}}{\Sigma_{\rm d}}\,\Omega^{-1}. \tag{4.16}$$

In the above, z is set to be the thickness of the dust layer, and Epstein's drag law (4.3) is used. The obtained growth time for dust is independent of the particle size. The growth time is estimated to be several $\times 10^4$yrs even at $\sim$100AU, which is much shorter than the disk lifetime. Thus, dust growth proceeds in a relatively short time.

(b) Dust settling

As dust particles grow, their settling velocity increases. When $t_{\text{grow}} \gtrsim h/|V_z(h)|$, dust settling is effective. The formation time of the thin dust layer at the disk midplane is $\sim 10\, t_{\text{grow}}$.

(c) Thickness of the dust layer in a turbulent disk

The thickness of the dust layer, h_d, is determined by the balance between the time scales of stirring by turbulence and settling.

$$\frac{h_d^2}{\nu_t} \sim \frac{h_d}{|V_z(h_d)|}. \tag{4.17}$$

Then, setting $\nu_t = \alpha c_s h$, we have

$$h_d = \sqrt{\frac{\alpha}{t_{\text{stop}}\Omega}}\, h. \tag{4.18}$$

In turbulent disks, the dust growth time of Eq. (4.16) is valid because the dust velocities accelerated by turbulence are comparable to the settling velocities.

(d) Dust radial drift

- The dust drift velocities increase as dust particles grow. For dust particles with $t_{\text{stop}}\Omega_{\text{K}} \simeq 1$, Eq. (4.10) gives

$$|V_R| \simeq \eta R \Omega \simeq 50\text{m/sec}. \tag{4.19}$$

- Radial drift time (for $t_{\text{stop}}\Omega_{\text{K}} \simeq 1$)

$$t_{\text{drift}} = \frac{R}{|V_R|} \simeq \frac{1}{\eta\Omega} \simeq 100\text{yr} \qquad (\text{at 1AU}). \tag{4.20}$$

Problem 6.

1. In Epstein's drag law (4.3), the drag force is roughly given by $F_{\text{drag}} \sim \pi a^2 \rho_g c_s \Delta v$. Derive this with an order estimate.
2. Derive the velocities of gas and dust of Eqs. (4.8)-(4.11).

4.3 Mechanics of dust sticking

- The origin of adhesion is intermolecular forces, which are van der Waals forces (<0.01eV in energy) for silicate grains or hydrogen bonds ($\sim$0.1eV) for ice grains.
- The energy of adhesion between particles can be expressed by the macroscopic surface energy γ [J/m^2] as

$$E_{\text{stick}} = -\,2\gamma\pi a^2, \tag{4.21}$$

where a is the radius of the contact surface between particles.

- Radius of the contact surface

  When two elastic spheres of radius R are in contact, the radius a of the contact surface (a circle) is given by (JKR theory)

$$a \simeq \left(\frac{14\gamma R^2}{\mathcal{E}}\right)^{1/3}, \tag{4.22}$$

  where $\mathcal{E}$ is the Young's modulus [Pa = N/m^2], which indicates the rigidity of the elastic particle. Let us derive this equation with an order estimate. Using the displacement, $u(\boldsymbol{r})$, of each part of the particle, the elastic energy is given by

$$E_{\rm elastic} \simeq \int (\text{stress}) \times \frac{du}{dx} dV \simeq \int \frac{1}{2}\mathcal{E}\left(\frac{du}{dx}\right)^2 dV \simeq 0.2\,\mathcal{E}a^5/R^2. \tag{4.23}$$

  The third equality of the above equation is approximately derived, by estimating du/dx as δ/a and using the geometric relation $\delta \sim a^2/R$, where δ is the displacement of the surface. The radius a is determined to minimize the total energy $E_{\rm stick} + E_{\rm elastic}$, and Eq. (4.22) is obtained.

- The binding energy due to adhesion can be obtained using the radius of Eq. (4.22) as

$$E_{\rm bond} = |E_{\rm stick} + E_{\rm elastic}| \simeq 20\left(\frac{\gamma^5 R^4}{\mathcal{E}^4}\right)^{1/3}. \tag{4.24}$$

- Velocity limit for sticking

  Equating the binding energy to the impact kinetic energy, and denoting the mass of the particle by m, the maximum velocity for sticking, $v_{\rm crit}$, is given by

$$v_{\rm crit} \sim \left(\frac{E_{\rm bond}}{m}\right)^{1/2} \sim \begin{cases} 3(R/0.1\mu\text{m})^{-5/6} & \text{m/sec} \quad \text{(ice)}, \\ 0.3(R/0.1\mu\text{m})^{-5/6} & \text{m/sec} \quad \text{(silicate)}. \end{cases} \tag{4.25}$$

  Equation (4.25) is a rough estimate. The critical velocity for sticking is one order of magnitude higher than the above estimate, according to more dedailed studies on dust collisions via numerical simulations or laboratory experiments.

Table 1: Constants in dust adhesion

	Surface energy γ [J/m^2]	Young's modulus $\mathcal{E}$ [GPa]
silicate (SiO_2)	0.03 (effective value)	50
ice	0.1	7

5 Planetesimal Formation

6 Kinetics of Planetesimals as Self-gravitating N-body System

7 Planetesimal Accretion

8 Collisional Disruption of Planetesimals

9 Planetary Migration via Disk-planet Interaction

10 Formation of Giant Planets via Gas Accretion

11 Future Problems in Planet Formation Theory